AN INTRODUCTION TO ELLIPTIC EQUATIONS ON $I\!R^N$

C. A. STUART

February 2, 1998

Contents

1 Introduction and notation

The purpose of these lectures is to establish some of the fundamental results concerning second order elliptic equations on $\mathbb{R}^N$ for $N \geq 2$. The discussion is focussed on three main topics, the regularity of weak solutions, the nature of the spectrum and the properties of eigenfunctions. Apart from their intrinsic interest, I think that a good understanding of this material in invaluable for further work on nonlinear elliptic equations on $\mathbb{R}^N$ and the form in which it is presented has been influenced by this belief. To make the exposition accessible to as broad an audience as possible, I have tried to give fairly complete proofs of all the results using a minimum amount of prerequisite knowledge, essentially the standard results about Sobolev spaces and some very basic functional analysis are assumed. For the same reason I have deliberately confined the references to a small number of widely available texts. In respecting these criteria within the limits of a short course of lectures some compromises had to be made, so the results obtained are not always the sharpest possible. Some remarks about extensions and improvements are made at appropriate places in the text.

The presentation of the results is divided into two parts. The first one is mainly devoted to the questions concerning the regularity of weak solutions. We establish these results in the simplest context, but, as we show in the final paragraph of that part, they can be used to obtain similar conclusions in much more general situations. The second part develops the spectral theory and completes the discussion of the existence of solutions which was treated from a very limited point of view in the first part.

The notation for function spaces is standard and, since most of the time, the functions are defined on $\mathbb{R}^N$, this domain of definition will be omitted. Thus,

- L^p denotes $L^p(\mathbb{R}^N)$ with the usual norm

$$|u|_p = \left\{ \int |u|^p \, dx \right\}^{\frac{1}{p}} \text{ for } 1 \leq p < \infty \text{ and } |u|_\infty = ess \sup_{x \in \mathbb{R}^N} |u(x)|$$

  where, in keeping with the above convention, an integral with no domain of integration is understood to extend over all $\mathbb{R}^N$.

- $W^{k,p}$ denotes the usual Sobolev space $W^{k,p}(\mathbb{R}^N)$ with the norm

$$\|u\|_{W^{k,p}} = \left\{ \sum_{0\leq|\alpha|\leq k} |D^\alpha u|_p^p \right\}^{\frac{1}{p}}$$

where $\alpha \in N^N$ is a multi-index and $D^\alpha u$ denotes a generalized partial derivative.

- $H^k = W^{k,2}$ with the above norm is a Hilbert space.

- $C = C(\mathbb{R}^N)$ denotes the vector space of all continuous functions on $\mathbb{R}^N$.

- $D = C_0^\infty$ denotes the vector space of all infinitely differentiable functions on $\mathbb{R}^N$ with compact support.

- L_{loc}^p denotes the vector space of all f such that $f \in L^p(\Omega)$ for every bounded open set $\Omega \subset \mathbb{R}^N$.

In general, these are spaces of real valued functions. In dealing with the Fourier transform, complex valued functions occur naturally and the corresponding spaces are denoted by italics. We recall the main properties of these spaces which will be used.

- $D = C_0^\infty$ is dense in L^p and $W^{k,p}$ for $1 \leq p < \infty$, Adams [10] Corollary 3.19.

- For $p > N, W^{1,p} \subset C \cap L^\infty$, there exists a constant K such that $|u|_\infty \leq K\|u\|_{W^{1,p}}$and $\lim_{|x|\to\infty} u(x) = 0$ for $u \in W^{1,p}$, Adams [10] Theorem 5.4 or Brézis [5] Théorème IX.12. It follows that $W^{1,p}$ is continuously imbedded in L^q for $p \leq q \leq \infty$ when $p > N$.

- For $p = N, W^{1,p}$ is continuously imbedded in L^q for $p \leq q < \infty$, Adams [10] Theorem 5.4 or Brézis [5] Corollaire IX.11.

- For $1 \leq p < N, W^{1,p}$ is continuously imbedded in L^q for $p \leq q < \frac{Np}{N-p}$, Adams [10] Theorem 5.4 or Brézis [5] Corollaire IX.10.

- For all N, $H^1(B(0,R))$ is compactly imbedded in $L^2(B(0,R))$ for any $R \in (0,\infty)$ where $B((0,R)) = \left\{x \in I\!R^N : |x| \leq R\right\}$, Adams [10] Theorem 6.2 or Brézis [5] Théorème IX.16.

- If $u \in H^1$, then $|u|$ and $u^+ \in H^1$ where $u^+ = \max\{u,0\}$. Furthermore, $|\nabla u|_2 = |\nabla |u||_2$ for all $u \in H^1$. See Gilbarg and Trudinger [6] Lemmas 7.6 and 7.7.

2 The generalized Helmholtz equation

In this part we discuss some properties of the generalized Helmholtz equation for $\lambda < 0$. The first step is to establish the existence and uniqueness of weak solutions in a simple way for a fairly general class of inhomogeneous terms. Then we introduce the notion of a fundamental solution and thereby prove the existence of classical solutions for smooth inhomogeneous terms of compact support. These solutions have an integral representation as the convolution of the fundamental solution with the inhomogeneous term. By exploiting Young's inequality for convolutions and approximation, we then establish some basic $W^{1,p}-$ estimates for weak solutions of the Helmholtz equation. Finally we recall some elementary properties of the Fourier transform which lead to H^2- estimates for weak solutions. Some applications of these results to more general linear and nonlinear elliptic equations are given.

2.1 Weak solutions

We consider solutions $u : I\!R^N \to I\!R$ of the problem

$$\begin{cases} -\Delta u(x) = \lambda u(x) + f(x) \\ \qquad u \in H^1 \end{cases} \tag{1}$$

where, as will be justified by the subsequent regularity theory, the condition $u \in H^1$ plays the role of a boundary condition in (1) well as providing a basis for the sense in which the differential equation is satisfied.

Definition 2.1 *Given $\lambda \in I\!R$ and $f \in L^1_{loc}$, we call u a weak solution of (1) $\Leftrightarrow u \in H^1$ and*

$$\int \nabla u \cdot \nabla v dx = \int \{\lambda u + f\} v dx \quad \textit{for all } v \in D.$$

(Observe that the integrals are finite for functions with the stated properties.)

A useful result about the existence of weak solutions in this sense can be deduced directly from the Sobolev inequalities and a direct application of the Riesz representation theorem for bounded linear functionals on a Hilbert space which we now recall.

Let $(H, \langle\cdot,\cdot\rangle)$ be a real Hilbert space whose topological dual is denoted by H^*. For every $F \in H^*$, there exists a unique element $u_F \in H$ such that

$$F(v) = \langle u_F, v\rangle \quad \text{ for all } v \in H.$$

Furthermore, $\|u_F\|_H = \|F\|_{H^*}$ and the mapping $F \longmapsto u_F$ is an isomorphism, Weidmann [1] or Gilbarg and Trudinger [6], Theorem 4.8.

Lemma 2.2 *Let $f \in L^p$ for some $p \in \left[\frac{2N}{N+2}, 2\right]$ with $p > 1$ when $N = 2$. Then, for all $v \in H^1, fv \in L^1$, and setting*

$$F(v) = \int fv dx \quad \text{ for } v \in H^1,$$

we have that $F \in (H^1)^* = H^{-1}$ and there is a constant, $C(N,p)$, such that

$$\|F\|_{H^{-1}} \leq C(N,p)\,|f|_p\,.$$

Proof By the Sobolev inclusions, Gilbarg and Trudinger [6] or Brézis [5], we have that

$$H^1 \subset L^q \quad \text{ where } \begin{cases} 2 \leq q < \infty \text{ if } N = 2 \\ 2 \leq q \leq \frac{2N}{N-2} \text{ if N} \geq 3 \end{cases}$$

and there is a constant $K(N,q)$ such that

$$|v|_q \leq K(N,q)\,\|v\|_{H^1} \quad \text{ for all } v \in H^1$$

under the same restrictions on q.

By Hölder's inequality,

$$\left|\int fv dx\right| \leq |f|_p\,|v|_{p'} \quad \text{ where } \frac{1}{p} + \frac{1}{p'} = 1.$$

The restrictions on p in the statement of the lemma ensure that p' satisfies the restrictions on q required for the Sobolev inclusions and so we have that

$$|F(v)| \leq K(N,p')\,\|v\|_{H^1}\,|f|_p\,.$$

This proves the result.

Theorem 2.3 *Let $\lambda < 0$ and $f_i \in L^{p_i}$ where $p_i \in \left[\frac{2N}{N+2}, 2\right]$, with $p_i > 1$ if $N = 2$, for $i = 1, .. \quad k$. Then there is a unique weak solution u of (1) for λ and $f = \sum_{i=1}^k f_i$. Furthermore, setting $u = G_\lambda f$, we have that $u = \sum_{i=1}^k G_\lambda f_i$ and*

$$\|G_\lambda f\|_{H^1} \leq \frac{1}{\min\{1, |\lambda|\}} \sum_{i=1}^k C(N, p_i) |f_{i.}|_{p_i}.$$

Remark 2.1 *We shall see in Part 2 (Corollary 3.17) that this conclusion is false for $\lambda \geq 0$.*

Proof For $\lambda < 0$ and $u, v \in H^1$, we set

$$a_\lambda(u, v) = \int \nabla u.\nabla v + |\lambda| \, uvdx.$$

Then

$$\begin{aligned} |a_\lambda(u, v)| &\leq |\nabla u|_2 |\nabla v|_2 + |\lambda| |u|_2 |v|_2 \\ &\leq (1 + |\lambda|) \|u\|_{H^1} \|v\|_{H^1} \end{aligned}$$

and so $a_\lambda : H^1 \times H^1 \to I\!R$ is a bounded, symmetric, bilinear functional. Since

$$\min\{1, |\lambda|\} \|u\|_{H^1}^2 \leq a_\lambda(u, u) = \int |\nabla u|^2 + |\lambda| \, u^2 dx \leq \max\{1, |\lambda|\} \|u\|_{H^1}^2,$$

a_λ is a scalar product on H^1 which induces a norm, $a_\lambda(u, u)^{\frac{1}{2}}$, equivalent to the usual one, $\|u\|_{H^1} = a_{-1}(u, u)^{\frac{1}{2}}$.

Setting

$$F_i(v) = \int f_i v dx \quad \text{for } v \in H^1,$$

it follows from the lemma that $F_i \in H^{-1}$ for $i = 1, .. \quad k$. Thus $F = \sum_{i=1}^k F_i \in H^{-1}$ and, applying the Riesz representation theorem on $(H^1, a_\lambda(\cdot, \cdot))$, there is a unique element $u \in H^1$ such that

$$a_\lambda(u, v) = F(v) \quad \text{for all } v \in H^1.$$

By the definition of a_λ this is a weak solution of (1) and, from the uniqueness, it follows easily that

$$G_\lambda f = \sum_{i=1}^k G_\lambda f_i.$$

Furthermore,

$$\begin{aligned}\min\{1,|\lambda|\}\,\|u\|_{H^1}^2 &\le a_\lambda(u,u)=F(u)=\sum_{i=1}^{k}F_i(u)\\ &\le \sum_{i=1}^{k}C(N,p_i)\,|f_i|_{p_i}\,\|u\|_{H^1}\end{aligned}$$

which establishes the last part of the theorem.

2.2 The fundamental solution

Radially symmetric non-trivial solutions of the homogeneous Helmholtz equation can be expressed in term of Bessel functions. Setting $\lambda=-k^2$ with $k>0$ and $f\equiv 0$, the Helmholtz equation becomes

$$\Delta u(x)-k^2u(x)=0 \quad \text{for } x\in \mathbb{R}^N \tag{2}$$

and for a radially symmetric solution, $u(x)=z(r)$ where $r=|x|$, this is equivalent to the differential equation

$$z''(r)+\frac{N-1}{r}z'(r)-k^2z(r)=0 \quad \text{for } r>0 \tag{3}$$

where the prime denotes differentiation with respect to r. Setting

$$z(r)=(kr)^{1-\frac{N}{2}}w(kr),$$

the equation (3) is transformed to the following Bessel equation for w,

$$w''(r)+\frac{1}{r}w'(r)-\left\{1+\frac{v^2}{r^2}\right\}w(r)=0 \quad \text{for } r>0, \tag{4}$$

where $v=\frac{N}{2}-1$. Two linearly independent solutions of (4) are given by the modified Bessel functions of order v of the first and second kind which are usually denoted by I_v and K_v respectively. The function I_v has a finite limit as $r\to 0$ but grows exponentially as $r\to\infty$, whereas K_v decays exponentially to zero as $r\to\infty$ but has a singularity at $r=0$. The latter is sometimes called MacDonald's function and it can be used to define the fundamental solution of (2) for $\lambda=-k^2$ and thereby to give an integral representation for solutions of the inhomogeneous Helmholtz equation. To this end we recall the relevant properties of K_v. (See Lebedev [3], particularly Sections 5.7 and 5.16.4.)

- For all $v \geq 0, K_v \in C^\infty((0,\infty))$ and the recurrence relation
$$\left\{r^{-v}K_v(r)\right\}' = -r^{-v}K_{v+1}(r) \quad \text{for } r > 0 \tag{5}$$
is valid.

- As $r \to 0$,
$$\begin{aligned} \frac{K_0(r)}{-\ln r} &\to 1 \quad \text{and} \\ r^v K_v(r) &\to l_v \quad \text{where } l_v > 0 \quad \text{for } v > 0, \\ r^{v+1}K_v'(r) &\to -L_v \quad \text{where } L_v > 0 \quad \text{for all } v \geq 0. \end{aligned} \tag{6}$$
In fact, $L_0 = 1$ and for $v > 0$,
$$\begin{aligned} l_v &= 2^{\nu-1}\Gamma(v) \text{ and } L_v = v l_v \\ &\quad \text{where } \Gamma \text{ denotes the Gamma function.} \end{aligned}$$

- As $r \to \infty$,
$$\begin{aligned} e^r K_v(r) &\to 0 \quad \text{and} \\ e^r K_v'(r) &\to 0 \quad \text{for all } v \geq 0. \end{aligned} \tag{7}$$

- For all $v \geq 0$ and $r > 0$, $K_v(r) > 0$.

We can now define the fundamental solution for the Helmholtz equation on $\mathbb{R}^N$ as follows. We begin with the case $\lambda = -1$ since the more general case $\lambda < 0$ can be reduced to this by simply rescaling the independent variable. Define the function $E_N : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ by
$$\begin{aligned} E_N(x) &= \gamma_N \, |x|^{1-\frac{N}{2}} K_{\frac{N}{2}-1}(|x|) \quad \text{for } x \neq 0 \quad \text{where} \\ \gamma_N &= \frac{1}{\omega_N l_{\frac{N}{2}}}. \end{aligned}$$
Here ω_N is the $(N-1)-$dimensional measure of the unit sphere in $\mathbb{R}^N$ and $l_{\frac{N}{2}}$ the limit defined in (6). Since
$$\begin{aligned} \omega_N &= \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \text{ and } l_{\frac{N}{2}} = 2^{\frac{N}{2}-1}\Gamma(\frac{N}{2}), \text{ it follows that} \\ \gamma_N &= (2\pi)^{-\frac{N}{2}}. \end{aligned}$$

The fundamental solution for the Helmholtz equation with $\lambda = -k^2$ on $I\!R^N$ is then the function $E_N^k : I\!R^N \setminus \{0\} \to I\!R$ defined by

$$E_N^k(x) = k^{N-2} E_N(kx) \quad \text{for } x \neq 0.$$

From the properties of the modified Bessel functions K_υ we deduce the following properties of the fundamental solutions E_N^k for $k > 0$.

- $E_N^k \in C^\infty(I\!R^N \setminus \{0\})$ and
$$\Delta E_N^k(x) - k^2 E_N^k(x) = 0 \quad \text{for } x \neq 0. \tag{8}$$
- $E_N^k(x) > 0 \quad$ for $x \neq 0$.
- As $|x| \to 0$,
$$\begin{aligned} \frac{E_2^k(x)}{-\ln(k\,|x|)} &\to \frac{1}{2\pi l_1} \\ |x|^{N-2} E_N^k(x) &\to \gamma_N l_{\frac{N}{2}-1} > 0 \quad \text{for } N \geq 3 \quad \text{and} \\ |x|^{N-1} \left|\nabla E_N^k(x)\right| &\to \gamma_N l_{\frac{N}{2}} > 0 \quad \text{for } N \geq 2. \end{aligned}$$
- As $|x| \to \infty$,
$$e^{k|x|} E_N^k(x) \to 0 \text{ and } e^{k|x|} \left|\nabla E_N^k(x)\right| \to 0.$$

From the last two properties it follows that

$$\begin{aligned} E_2^k &\in L^p \quad \text{for } 1 \leq p < \infty \\ E_N^k &\in L^p \quad \text{for } 1 \leq p < \frac{N}{N-2} \quad \text{for } N \geq 3 \\ \partial_i E_N^k &\in L^p \quad \text{for } 1 \leq p < \frac{N}{N-1} \quad \text{for } N \geq 2 \quad \text{and } i = 1,.. \quad N. \end{aligned} \tag{9}$$

Using this information about E_N^k we can now express solutions of the inhomogeneous Helmholtz equation as the convolution of the fundamental solution and the inhomogeneous term. We begin by recalling an elementary result about convolution.

Lemma 2.4 *Let $f \in C_0$ and let $g \in L^1$. Then for all $x \in \mathbb{R}^N$,*

$$y \longmapsto f(x-y)g(y) \in L^1$$

and, setting

$$f \star g(x) = \int f(x-y)g(y)dy \quad \text{for } x \in \mathbb{R}^N,$$

we have that $f \star g \in C$ with $\lim_{|x|\to\infty} f \star g(x) = 0, |f \star g|_\infty \leq |f|_\infty |g|_1$ and

$$f \star g(x) = \int f(y)g(x-y)dy.$$

Proof For all $x \in \mathbb{R}^N$,

$$|f(x-y)g(y)| \leq |f|_\infty |g(y)| \quad \text{for almost all } y \in \mathbb{R}^N$$

and so

$$\int |f(x-y)g(y)|\, dy \leq |f|_\infty |g|_1 .$$

By the Dominated Convergence Theorem it follows that

$$\lim_{|x|\to\infty} \int f(x-y)g(y)dy = 0.$$

For $x, z \in \mathbb{R}^N$,

$$\begin{aligned} |f \star g(x) - f \star g(z)| &\leq \int |f(x-y) - f(z-y)|\, |g(y)|\, dy \\ &\leq \max_{y \in \mathbb{R}^N} |f(x-y) - f(z-y)|\, |g|_1 . \end{aligned}$$

Since f is uniformly continuous on $\mathbb{R}^N$, this shows that $f \star g$ is continuous on $\mathbb{R}^N$. Finally, setting $\tilde{y} = x - y$, we obtain

$$\int f(x-y)g(y)dy = \int f(\tilde{y})g(x-\tilde{y})d\tilde{y}.$$

We can now use the fundamental solution to generate classical solutions of the Helmholtz equation.

Theorem 2.5 *Given $k > 0$ and $f \in C_0^1(I\!R^N)$, we set*

$$T_k f(x) = f \star E_N^k(x) \quad \text{for } x \in I\!R^N. \tag{10}$$

Then $T_k f \in C^2(I\!R^N), \lim_{|x|\to\infty} T_k f(x) = 0$ and $u = T_k f$ satisfies the Helmholtz equation

$$-\Delta u(x) = \lambda u(x) + f(x) \text{ on } I\!R^N \tag{11}$$

where $\lambda = -k^2$. Furthermore, for all $x \in I\!R^N$, the following formulae are valid for $i, j = 1, .. \quad N$,

$$\begin{aligned}
T_k f(x) &= k^{N-2} \int f(x-y) E_N(ky) dy = k^{N-2} \int E_N(k(x-y)) f(y) dy \\
\partial_i T_k f(x) &= k^{N-2} \int \partial_i f(x-y) E_N(ky) dy = k^{N-1} \int \partial_i E_N(k(x-y)) f(y) dy \\
\partial_j \partial_i T_k f(x) &= k \int \partial_i f(x-y) \partial_j E_N(ky) dy.
\end{aligned}$$

Remark 2.2 *Noting that*

$$T_k f(x) = T_1 f_k(kx) \text{ where } f_k(y) = k^{-2} f(\frac{y}{k}),$$

we see that, by a change of scale, it is enough to treat the case $k = 1$.

Remark 2.3 *Further properties of $T_k f$ will be established in the next section. Here we simply note that, under the hypotheses of the above theorem, $\partial_i T_k f(x) \to 0$ as $|x| \to \infty$ for $i = 1, .. \quad N$. This follows from Lemma 4. Results concerning the integrability of $T_k f$ are given in the next section where it will also be shown that $T_k f$ coincides with the unique weak solution of (1) established in Theorem 2.3.*

Remark 2.4 *Setting*

$$\Psi_N^k(x) = (k\,|x|)^{1-\frac{N}{2}} I_{\frac{N}{2}-1}(k\,|x|) \quad \text{for } x \neq 0, \tag{12}$$

it follows from the (2) to (4) that $c\Psi_N^k \in C^2(R^N)$ and satisfies (2) for $\lambda = -k^2$ for all $c \in I\!R$. Consequently, $T_k f + c\Psi_N^k$ satisfies (10) for all constants $c \in I\!R$. However $\Psi_N^k(x)$ grows exponentially as $|x| \to \infty$.

Remark 2.5 *Anticipating the introduction of the Fourier transform in the next section, we note that the function E_N could be defined as the inverse Fourier transform of $Q(\xi) = 1/(1+|\xi|^2)$ for $\xi \in \mathbb{R}^N$. In this context, the convolution $f \star E_N$ is often called a Bessel potential of order 2. However, $Q \in L^p \iff p > \frac{N}{2}$ and so, in general, this approach requires the inverse Fourier transform to be defined on L^p for all $p \in (1,\infty)$. Although this is possible, we prefer to avoid this more sophisticated discussion by appealing directly to the properties of the modified Bessel functions summarized above. The explicit formulae connecting the two approaches follow from Titchmarsh [8] (7.11.12) and (8.19.5) together with the expression for the Fourier transform of the radially symmetric function Q as a Hankel transform.*

Proof We take $k = 1$ and simplify the notation by setting $T_1 = T$.

Since f and $\partial_i f \in C_0(\mathbb{R}^N)$ and E_N and $\partial_j E_N \in L^1(\mathbb{R}^N)$, it follows from Lemma 2.4 that the convolutions

$$Tf = f \star E_N, \partial_i f \star E_N, f \star \partial_i E_N \text{ and } \partial_i f \star \partial_j E_N$$

are defined and are continuous on $\mathbb{R}^N$. They all tend to zero as $|x| \to \infty$. Hence to prove the theorem it is sufficient to establish the following statements.

1. $\partial_i Tf$ exists and $\partial_i Tf = \partial_i f \star E_N$.
2. $\partial_i f \star E_N = f \star \partial_i E_N$.
3. $\partial_j \partial_i Tf$ exists and $\partial_j \partial_i Tf = \partial_j f \star \partial_i E_N$.
4. $-\Delta Tf + Tf = f$ on $\mathbb{R}^N$.

(1) Let e_i be an element of the usual basis for $\mathbb{R}^N$ and h a non-zero real number. Then

$$\frac{Tf(x+he_i) - Tf(x)}{h} = \int \left\{ \frac{f(x+he_i-y) - f(x-y)}{h} \right\} E_N(y)dy$$

and

$$\lim_{h\to 0} \frac{f(x+he_i-y) - f(x-y)}{h} = \partial_i f(x-y).$$

Also

$$\begin{aligned}
\left|\frac{f(x+he_i-y)-f(x-y)}{h}\right| &= \left|\frac{1}{h}\int_0^1 \frac{d}{dt} f(x+the_i-y)dt\right| \\
&= \left|\int_0^1 \partial_i f(x+the_i-y)dt\right| \\
&\leq \max_{z\in \mathbb{R}^N} |\partial_i f(z)| = |\partial_i f|_\infty .
\end{aligned}$$

Hence by the dominated convergence theorem,

$$\lim_{h\to 0} \frac{Tf(x+he_i)-Tf(x)}{h} = \int \partial_i f(x-y)E_N(y)dy.$$

(2) For $i=1,.. \quad N$,

$$\begin{aligned}
\partial_i f \star E_N(x) &= \lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon} \partial_i f(x-y)E_N(y)dy \\
&= \lim_{\varepsilon\to 0}\left\{-\int_{|y|\geq\varepsilon} \frac{\partial}{\partial y_i} f(x-y)E_N(y)dy\right\} \\
&= \lim_{\varepsilon\to 0}\left\{\int_{|y|=\varepsilon} \frac{y_i}{|y|} f(x-y)E_N(y)dy + \int_{|y|\geq\varepsilon} f(x-y)\partial_i E_N(y)dy\right\}.
\end{aligned}$$

Now

$$\begin{aligned}
&\left|\int_{|y|=\varepsilon} \frac{y_i}{|y|} f(x-y)E_N(y)dy\right| \leq |f|_\infty \int_{|y|=\varepsilon} |E_N(y)|\, dy \\
= \;& |f|_\infty \left\{\gamma_N \varepsilon^{1-\frac{N}{2}} K_{\frac{N}{2}-1}(\varepsilon)\int_{|y|=\varepsilon} dy\right\} = |f|_\infty \left\{\gamma_N \varepsilon^{1-\frac{N}{2}} K_{\frac{N}{2}-1}(\varepsilon)\right\}\omega_N \varepsilon^{N-1} \\
= \;& |f|_\infty\, \gamma_N \omega_N \varepsilon^{\frac{N}{2}} K_{\frac{N}{2}-1}(\varepsilon).
\end{aligned}$$

Hence

$$\lim_{\varepsilon\to 0}\int_{|y|=\varepsilon} \frac{y_i}{|y|} f(x-y)E_N(y)dy = 0$$

and

$$\partial_i f \star E_N(x) = \lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon} f(x-y)\partial_i E_N(y)dy = f\star \partial_i E_N.$$

(3) Repeat the proof of (1) with E_N replaced by $\partial_i E_N$.

(4) For $x \in I\!\!R^N$,

$$\begin{aligned}
\Delta T f(x) &= \sum_{i=1}^{N} \int \partial_i f(x-y)\partial_i E_N(y)dy \\
&= \lim_{\varepsilon\to 0} -\sum_{i=1}^{N} \int_{|y|\geq\varepsilon} \frac{\partial}{\partial y_i} f(x-y)\partial_i E_N(y)dy \\
&= \lim_{\varepsilon\to 0} \sum_{i=1}^{N} \int_{|y|=\varepsilon} \frac{y_i}{|y|} f(x-y)\partial_i E_N(y)dy + \int_{|y|\geq\varepsilon} f(x-y)\Delta E_N(y)dy \\
&= \lim_{\varepsilon\to 0} \sum_{i=1}^{N} \int_{|y|=\varepsilon} \frac{y_i}{|y|} f(x-y)\partial_i E_N(y)dy + Tf(x)
\end{aligned}$$

since $\Delta E_N(y) = E_N(y)$ for all $y \neq 0$.

For $y \neq 0$,

$$\begin{aligned}
\partial_i E_N(y) &= \gamma_N \left\{ |y|^{1-\frac{N}{2}} K_{\frac{N}{2}-1}(|y|) \right\}' \frac{y_i}{|y|} \\
&= -\gamma_N \left\{ |y|^{1-\frac{N}{2}} K_{\frac{N}{2}}(|y|) \right\} \frac{y_i}{|y|} \text{ by (5)}
\end{aligned}$$

and so

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{|y|=\varepsilon} \frac{y_i}{|y|} f(x-y)\partial_i E_N(y)dy = -\gamma_N \varepsilon^{1-\frac{N}{2}} K_{\frac{N}{2}}(\varepsilon) \int_{|y|=\varepsilon} f(x-y)dy \\
= \; & -\gamma_N \varepsilon^{1-\frac{N}{2}} K_{\frac{N}{2}}(\varepsilon) \int_{|y|=\varepsilon} \{f(x-y) - f(x)\}\, dy - \gamma_N \varepsilon^{1-\frac{N}{2}} K_{\frac{N}{2}}(\varepsilon)\omega_N \varepsilon^{N-1} f(x)
\end{aligned}$$

where

$$\varepsilon^{\frac{N}{2}} K_{\frac{N}{2}}(\varepsilon) \to l_{\frac{N}{2}} \text{ as } \varepsilon \to 0$$

and

$$|f(x-y) - f(x)| \leq |\nabla f|_\infty |y| \text{ for all } y \in I\!\!R^N.$$

Hence

$$\Delta T f(x) = Tf(x) - \gamma_N l_{\frac{N}{2}} \omega_N f(x) = Tf(x) - f(x).$$

2.3 $W^{1,p}$–regularity

The aim is to obtain a sharper relationship between the regularity of the weak solutions of the Helmholtz equation and the properties of the inhomogeneous term f in (1). In the approach we propose the first step involves obtaining estimates for the representation $T_k f$ which was introduced in the previous section. These estimates are deduced from *Young's inequality for convolutions* which we now recall, Titchmarsh [8] Chapter IV, Lemma β.

Theorem 2.6 *Let $f \in L^p$ and $g \in L^q$ with $\frac{1}{p} + \frac{1}{q} \geq 1$. Then*

$$\int f(x-y)g(y)dy \quad \textit{converges for almost all } x \in I\!R^N$$

and defines an element of L^s where $\frac{1}{s} = \frac{1}{p} + \frac{1}{q} - 1$ (with $s = \infty$ when $\frac{1}{p} + \frac{1}{q} = 1$). Denoting this element by $f \star g$, we also have that $f \star g = g \star f$ and

$$|f \star g|_s \leq |f|_p \, |g|_q \, .$$

Remark 2.6 *Note that this definition coincides with the convolution introduced in the previous section where $p = \infty$ and $q = 1$.*

Remark 2.7 *Since p and $q \geq 1$, $\frac{1}{p} + \frac{1}{q} \leq 2$ and so $s \geq 1$.*

Proof This result can be established using Hölder's inequality and Fubini's Theorem. In the case where $\frac{1}{p} + \frac{1}{q} > 1$, we define s by $\frac{1}{s} = \frac{1}{p} + \frac{1}{q} - 1$ observe that

$$\begin{aligned} 1 - \frac{p}{s} &= p(1 - \frac{1}{q}) \geq 0 \quad \text{and} \\ 1 - \frac{q}{s} &= q(1 - \frac{1}{p}) \geq 0. \end{aligned}$$

Now

$$|f(x-y)|\,|g(y)| = \{|f(x-y)|^p \, |g(y)|^q\}^{\frac{1}{s}} \, |f(x-y)|^{1-\frac{p}{s}} \, |g(y)|^{1-\frac{q}{s}}$$

and setting

$$\alpha = \frac{p}{1-\frac{p}{s}} \text{ and } \beta = \frac{q}{1-\frac{q}{s}} \text{ with } \alpha = \infty \text{ if } p = s \text{ and } \beta = \infty \text{ if } q = s,$$

we have that

$$\frac{1}{s}+\frac{1}{\alpha}+\frac{1}{\beta}=1$$

and so from Hölder's inequality with three factors we obtain

$$\begin{aligned}
&\int |f(x-y)|\,|g(y)|\,dy \\
\leq\ & \left\{\int |f(x-y)|^p\,|g(y)|^q\,dy\right\}^{\frac{1}{s}}\left\{\int |f(x-y)|^p\,dy\right\}^{\frac{1}{\alpha}}\left\{\int |g(y)|^q\,dy\right\}^{\frac{1}{\beta}} \\
=\ & \left\{\int |f(x-y)|^p\,|g(y)|^q\,dy\right\}^{\frac{1}{s}} |f|_p^{\frac{p}{\alpha}}\,|g|_q^{\frac{q}{\beta}} \qquad \text{for almost all } x\in I\!\!R^N.
\end{aligned}$$

Hence

$$\begin{aligned}
\int\left\{\int |f(x-y)|\,|g(y)|\,dy\right\}^s dx &\leq |f|_p^{\frac{sp}{\alpha}}\,|g|_q^{\frac{sq}{\beta}}\int\int |f(x-y)|^p\,|g(y)|^q\,dydx \\
&= |f|_p^{\frac{sp}{\alpha}}\,|g|_q^{\frac{sq}{\beta}}\,|f|_p^p\,|g|_q^q \qquad \text{by Fubini's Theorem} \\
&= |f|_p^s\,|g|_q^s\,.
\end{aligned}$$

It follows that

$$\int f(x-y)g(y)dy\in L^s \text{ and that } |f\star g|_s\leq |f|_p\,|g|_q\,.$$

In the case where $\frac{1}{p}+\frac{1}{q}=1$,

$$\int |f(x-y)|\,|g(y)|\,dy\leq\left\{\int |f(x-y)|^p\,dy\right\}^{\frac{1}{p}}\left\{\int |g(y)|^q\,dy\right\}^{\frac{1}{q}}=|f|_p\,|g|_q$$

and so

$$|f\star g|_\infty\leq |f|_p\,|g|_q\,.$$

Recalling the properties (9) of the fundamental solution E_N^k for $k>0$, Young's inequality shows that the convolution $f\star E_N^k$ defines an element of L^s whenever $f\in L^p$ subject to the following restrictions,

$$\left\{\begin{array}{ll} p\leq s\leq\infty & \text{if } p>\frac{N}{2} \\ p\leq s<\infty & \text{if } p=\frac{N}{2} \\ p\leq s<\frac{Np}{N-2p} & \text{if } 1\leq p<\frac{N}{2} \end{array}\right. . \tag{13}$$

Moreover, setting $T_k f=f\star E_N^k$,, we see that

$$T_k: L^p\to L^s \text{ is a bounded linear operator} \tag{14}$$

under the restrictions (13).

Remark 2.8 *When $f \in C_0^1$ this definition of $T_k f$ coincides with that introduced in Theorem 2.5).*

Referring again to (9),for $i = 1, ..\ \ N$, the convolution $f \star \partial_i E_N^k$ defines an element of L^s whenever $f \in L^p$ subject to the following restrictions,

$$\begin{cases} p \leq s \leq \infty & \text{if } p > N \\ p \leq s < \infty & \text{if } p = N \\ p \leq s < \frac{Np}{N-p} & \text{if } 1 \leq p < N \end{cases} \tag{15}$$

and setting, $S_k^i f = f \star \partial_i E_N^k$ for $i = 1, ..\ \ N$, we see that

$$S_k^i : L^p \to L^s \text{ is a bounded linear operator} \tag{16}$$

under the restrictions (15).

We now establish the relationship between $T_k f$ and the weak solution of (1).

Theorem 2.7 *Let $\lambda = -k^2$ where $k > 0$ and let $f \in L^p$ where $p \in (\frac{2N}{N+2}, 2]$. Then $T_k f \in H^1$ and $\partial_i T_k f = S_k^i f$. Furthermore $T_k f$ is a weak solution of (1) and so by Theorem 2.3, $T_k f = G_\lambda f$.*

Proof Let $\{f_n\} \subset C_0^1$ be a sequence such that $|f_n - f|_p \to 0$ as $n \to \infty$. Since $p > \frac{2N}{N+2}$, we have that $\frac{2N}{N-2} < \frac{Np}{N-2p}$ when $N > 2$ and $p < \frac{N}{2}$. From (14) it follows that

$$T_k f_n \text{ and } T_k f \in L^s \text{ and that } |T_k f_n - T_k f|_s \to 0 \text{ as } n \to \infty$$

provided that $p \leq s \leq \frac{2N}{N-2}$ for $N \geq 3$ and $p \leq s < \infty$ for $N = 2$. Similarly $2 < \frac{Np}{N-p}$ when $p < N$. From (16), it follows that

$$S_k^i f_n \text{ and } S_k^i f \in L^s \text{ and that } \left|S_k^i f_n - S_k^i f\right|_s \to 0 \text{ as } n \to \infty$$

provided that $p \leq s \leq 2$.

By Theorem 2.5 we know that $T_k f_n \in C^2$ and that $\partial_i T_k f_n = S_k^i f_n$ for $i = 1, ..\ \ N$. Putting $s = 2$ in the preceding statements we deduce that

$T_k f \in H^1$ with $\partial_i T_k f = S_k^i f$ for $i = 1, .. \quad N$. Furthermore, setting $w = T_k f$, for any $v \in D$, we have

$$\begin{aligned}
& \int \nabla w . \nabla v - \{\lambda w + f\} v dx \\
= & -\int w \Delta v + \{\lambda w + f\} v dx \\
= & \lim_{n\to\infty} -\int T_k f_n \Delta v + \{\lambda T_k f_n + f_n\} v dx \\
= & \lim_{n\to\infty} -\int \Delta(T_k f_n) v + \left\{-k^2 T_k f_n + f_n\right\} v dx \\
= & \lim_{n\to\infty} -\int \{\Delta(T_k f_n) - k^2 T_k f_n + f_n\} v dx = 0
\end{aligned}$$

by Theorem 2.5. This proves that w is a weak solution of (1).

Having established this relationship between weak solutions and the convolution with the fundamental solution we now have a better understanding of the regularity of the weak solution.

Theorem 2.8 *Let $f \in L^p \cap L^q$ where $p \in (\frac{2N}{N+2}, 2]$ and $q \geq p$. Let u be a solution of (1) for f and some $\lambda \in \mathbb{R}$. Then $u \in H^1$ has the following additional properties.*

(i) $u \in W^{1,s}$ where

$$\begin{cases} p \leq s \leq \infty \text{ if } q > N \\ p \leq s < \infty \text{ if } q = N \\ p \leq s < \frac{Nq}{N-q} \text{ if } q < N \end{cases} \tag{17}$$

(ii) If $q > \frac{N}{2}$, then $u \in L^\infty \cap C$ and

$$\lim_{|x|\to\infty} u(x) = 0.$$

(iii) $u \in L^s$ where

$$\begin{cases} p \leq s < \infty \text{ if } q = \frac{N}{2} \\ p \leq s < \frac{Nq}{N-q} \text{ if } q < \frac{N}{2} \end{cases} \tag{18}$$

Remark 2.9 *Since $\frac{Nq}{N-q} > 2$ for $\frac{2N}{N+2} < q < N$, this result shows that $u \in W^{1,s}$ for some $s > 2$.*

Proof (i) For all $v \in D$,

$$\int \nabla u.\nabla v dx = \int \{\lambda u + f\} v dx = \int \{-u + g\} v dx$$

where $g = (\lambda+1)u + f$. Now, since $u \in H^1$, $(\lambda+1)u \in L^r$ for $2 \leq r < \infty$ if $N = 2$ and $2 \leq r < \frac{2N}{N-2}$ if $N \geq 3$. Thus u is a weak solution of $-\Delta u = -u+g$ and so, in the notation of Theorem (2.3), $u = G_{-1}(\lambda+1)u + G_{-1}f$. Then Theorem (2.7) shows that $u = T_1(\lambda+1)u + T_1 f$ and, using a bootstrap, argument to deal with the term $G_{-1}(\lambda+1)u$, the result now follows from (13) to (16).

(ii) By (i), $u \in W^{1,s}$ for some $s > N$ provided that $q > \frac{N}{2}$. For $s > N$, $W^{1,s} \subset C \cap L^\infty$ and $\lim_{|x|\to\infty} u(x) = 0$ for all $u \in W^{1,s}$.

(iii) This follows from (i) and the Sobolev inclusions, or directly from (13).

Theorem 2.9 *Let $f \in L^p \cap L^q$ where $p \in (\frac{2N}{N+2}, 2]$ and $q > \frac{N}{2}$. Suppose that $f \geq 0$ on $\mathbb{R}^N$ but $\not\equiv 0$. Then for every $\lambda < 0$, the unique weak solution, $u = G_\lambda f$, of (1) has the following properties.*

$$u \in C, \lim_{|x|\to\infty} u(x) = 0 \text{ and } u(x) > 0 \text{ for all } x \in \mathbb{R}^N.$$

Proof By Theorem 2.9, $u \in C$ and $\lim_{|x|\to\infty} u(x) = 0$. Furthermore, by Theorem 2.8,

$$u = T_k f = \int f(x-y) E_N^k(y) dy$$

where $\lambda = -k^2$ with $k > 0$. Recalling that $E_N^k(y) > 0$ for all $y \neq 0$ and that $f \geq 0$ on $\mathbb{R}^N$ but $\not\equiv 0$, we see that $u(x) > 0$ for all $x \in \mathbb{R}^N$.

2.4 H^2–regularity

The main result of this section shows that a weak solution of (1) belongs to H^2 whenever $f \in L^2$. This can be deduced from some standard properties of the Fourier transform which we now recall, Weidmann [1] or Yosida [7].

In dealing with the Fourier transform we must use complex-valued functions. The corresponding function spaces will be distinguished from the real counterparts by the use of italics. Thus $\mathcal{L}^p = L^p(\mathbb{R}^N, \mathbb{C})$ and $L^p =$

$L^p(I\!R^N, I\!R)$ etc.. The Schwartz space of smooth rapidly decreasing functions will be denoted by

$$\begin{aligned} \mathcal{S} &= \mathcal{S}(I\!R^N, \mathbb{C}) \\ &= \left\{ v \in \mathcal{C}^\infty : |x|^j \, D^\alpha v(x) \in \mathcal{L}^\infty \text{ for all } j \in N \text{ and all multi-indices } \alpha \in N^N \right\} . \end{aligned}$$

For $v \in \mathcal{S}$ (or more generally $v \in \mathcal{L}^1$), its Fourier transform $\widehat{v}$ is defined by

$$\widehat{v}(\xi) = (2\pi)^{-\frac{N}{2}} \int v(x) e^{-i\xi . x} dx \quad \text{ for all } \xi \in I\!R^N.$$

It has the following well-known properties.

- $\widehat{v} \in \mathcal{S}$ for all $v \in \mathcal{S}$ and (Parseval's identity)

$$\int v\overline{w} dx = \int \widehat{v}\overline{\widehat{w}} dx \quad \text{ for all } v, w \in \mathcal{S}.$$

- $\widehat{(\widehat{v})}(x) = v(-x)$ for all $x \in I\!R^N$ and for all $v \in \mathcal{S}$. Thus $\mathcal{S}$ is mapped bijectively onto $\mathcal{S}$, and defining

$$\check{v}(\xi) = (2\pi)^{-\frac{N}{2}} \int v(x) e^{i\xi . x} dx \quad \text{ for all } \xi \in I\!R^N,$$

  we see that $v \longmapsto \check{v}$ is the inverse of the transformation $v \longmapsto \hat{v}$.

- For $v \in \mathcal{S}$, $\partial_j v \in \mathcal{S}$ for all $j = 1, .. \quad N$ and

$$\widehat{\partial_j v}(\xi) = i\xi_j \widehat{v}(\xi) \quad \text{ for all } \xi \in I\!R^N,$$

  and, more generally,

$$\widehat{D^\alpha v}(\xi) = (i\xi)^\alpha \widehat{v}(\xi) \quad \text{ for all } \xi \in I\!R^N.$$

- Since $\mathcal{S}$ is dense in $\mathcal{L}^2$, the Fourier transform has a unique continuous extension to $\mathcal{L}^2$ which we continue to denote by $v \longmapsto \hat{v}$. By the Parseval identity we have that

$$|v|_2 = |\widehat{v}|_2 \text{ for all } v \in \mathcal{S}$$

  and so, since $\mathcal{S}$ is dense in $\mathcal{L}^2$, the Fourier transform has a unique continuous extension to $\mathcal{L}^2$ which we continue to denote by $v \longmapsto \hat{v}$. It is an isometric isomorphism of $\mathcal{L}^2$ onto itself and Parseval's identity remains valid for all $v, w \in \mathcal{L}^2$.

The following result plays a crucial role in the discussion of H^2-regularity.

Theorem 2.10 *For all* $v \in \mathcal{H}^2$,

$$\widehat{\partial_j \partial_k v} = -\xi_j \xi_k \widehat{v} \text{ for all } j, k = 1, .. \quad N.$$

Furthermore,

$$\mathcal{H}^2 = \left\{ v \in \mathcal{L}^2 : |\xi|^2 \widehat{v}(\xi) \in \mathcal{L}^2 \right\}$$

and

$$\|v\|_2 = \left\{ \int (1 + |\xi|^4) |\widehat{v}(\xi)|^2 \, d\xi \right\}^{\frac{1}{2}}$$

defines a norm on $\mathcal{H}^2$ *which is equivalent to the original norm*

$$\|v\|_{H^2} = \left\{ |v|_2^2 + \sum_{j=1}^{N} |\partial_j v|_2^2 + \sum_{j,k=1}^{N} |\partial_j \partial_k v|_2^2 \right\}^{\frac{1}{2}}.$$

Proof If $v \in \mathcal{H}^2$, there is a sequence $\{v_n\} \subset \mathcal{D} \subset \mathcal{S}$ such that $\|v_n - v\|_{H^2} \to 0$. Now

$$|\widehat{v_n} - \widehat{v}|_2 = |v_n - v|_2 \to 0$$

and

$$\begin{aligned} \int |\xi|^4 |\widehat{v_n} - \widehat{v_m}|^2 \, d\xi &= \int |\xi|^2 (\widehat{v_n} - \widehat{v_m}) \, |\xi|^2 \overline{(\widehat{v_n} - \widehat{v_m})} d\xi \\ &= \int |\Delta(v_n - v_m)(\xi)|^2 \, d\xi \leq C \, \|v_n - v_m\|_{H^2} . \end{aligned}$$

Hence $\left\{ |\xi|^2 \widehat{v_n} \right\}$ is a Cauchy sequence in $\mathcal{L}^2$ and we may suppose that $|\xi|^2 \widehat{v_n} \to z$ in $\mathcal{L}^2$.

For any $R > 0$,

$$\begin{aligned} \int_{|\xi| \leq R} |\xi|^4 |\widehat{v}|^2 \, d\xi &= \lim_{n \to \infty} \int_{|\xi| \leq R} |\xi|^4 |\widehat{v_n}|^2 \, d\xi \\ &= \int_{|\xi| \leq R} |z|^2 \, d\xi \leq |z|_2^2 , \end{aligned}$$

showing that $|\xi|^2 \widehat{v} \in \mathcal{L}^2$. This proves that

$$\mathcal{H}^2 \subset \left\{ v \in \mathcal{L}^2 : |\xi|^2 \widehat{v}(\xi) \in \mathcal{L}^2 \right\}.$$

The first identity, which we know holds in $\mathcal{S}$, also follows by this approximation.

Conversely, suppose that $v \in \mathcal{L}^2$ and that $|\xi|^2 \hat{v}(\xi) \in \mathcal{L}^2$. Then $(1+ |\xi|^2)\hat{v}(\xi) \in \mathcal{L}^2$ which means that there is a sequence $\{\varphi_n\}$ in $\mathcal{D}$ such that $\varphi_n \to (1+ |\xi|^2)\hat{v}(\xi)$ in $\mathcal{L}^2$. Setting

$$w_n(\xi) = \frac{\varphi_n(\xi)}{1+|\xi|^2}$$

we have that $w_n \in \mathcal{D}$ and, for all $j,k = 1, .. \quad N$, $\{w_n\} \subset \mathcal{S}$ and $\{\xi_j \xi_k w_n\} \subset \mathcal{S}$ are Cauchy sequences in $\mathcal{L}^2$. Thus $\{\check{w}_n\} \subset \mathcal{S}$ and $\{\partial_j \partial_k \check{w}_n\} \subset \mathcal{S}$ are also Cauchy sequences in $\mathcal{L}^2$. This means that $\{\check{w}_n\} \subset \mathcal{S}$ is a Cauchy sequence in $\mathcal{H}^2$ and so there is an element h of $\mathcal{H}^2$ such that $\check{w}_n \to h$ in $\mathcal{H}^2$. In particular, this implies that $\check{w}_n - \Delta \check{w}_n \to h - \Delta h$ in $\mathcal{L}^2$ and so, by the first identity, $(1+|\xi|^2)w_n \to (1+|\xi|^2)\hat{h}$ in $\mathcal{L}^2$. But, $(1+|\xi|^2)w_n = \varphi_n \to (1+|\xi|^2)\hat{v}$ in $\mathcal{L}^2$. Hence $\hat{v} = \hat{h}$, from which it follows that $v = h \in \mathcal{H}^2$, completing the proof.

The equivalence of the norms follows from the identity.

Using these properties of the Fourier transform we obtain the following result about generalized derivatives.

Lemma 2.11 *Let $v, w \in L^2$ be such that*

$$\int v \Delta z dx = \int w z dx \quad \textit{for all } z \in D.$$

Then $v \in H^2$,

$$-|\xi|^2 \hat{v}(\xi) = \hat{w}(\xi) \textit{ for almost all } \xi \in I\!R^N$$

and $\Delta v = w$.

Proof Since D is dense in H^2, we have that

$$\int v \Delta z dx = \int w z dx \quad \text{for all } z \in H^2.$$

But for $\varphi \in \mathcal{S}$, the real and imaginary parts of φ belong to H^2 and so

$$\int v \overline{\Delta \varphi} dx = \int w \overline{\varphi} dx \quad \text{for all } \varphi \in \mathcal{S}.$$

Furthermore,
$$\int v\overline{\Delta\varphi}dx = \int \widehat{v}\overline{\widehat{\Delta\varphi}}dx = -\int \widehat{v}\left|\xi\right|^2\overline{\widehat{\varphi}}d\xi$$
and
$$\int w\overline{\varphi}dx = \int \widehat{w}\overline{\widehat{\varphi}}d\xi.$$
Thus we see that
$$-\int \widehat{v}\left|\xi\right|^2\overline{\widehat{\varphi}}d\xi = \int \widehat{w}\overline{\widehat{\varphi}}d\xi \quad \text{ for all } \varphi \in \mathcal{S}.$$
Since $\overline{\widehat{\mathcal{S}}} = \mathcal{S}$, this means that
$$-\int \widehat{v}\left|\xi\right|^2\eta d\xi = \int \widehat{w}\eta d\xi \text{ for all } \eta \in \mathcal{S}.$$
In particular,
$$\left|\int \widehat{v}\left|\xi\right|^2\eta d\xi\right| \leq \left|\widehat{w}\right|_2 \left|\eta\right|_2 \text{ for all } \eta \in \mathcal{S}$$
and since $\mathcal{S}$ is dense in $\mathcal{L}^2$, it follows that
$$\left|\xi\right|^2\widehat{v}(\xi) \in \mathcal{L}^2 \text{ and } -\left|\xi\right|^2\widehat{v}(\xi) = \widehat{w}(\xi) \text{ for almost all } \xi \in I\!\!R^N.$$
Thus $v \in \left\{v \in \mathcal{L}^2 : \left|\xi\right|^2\widehat{v}(\xi) \in \mathcal{L}^2\right\} = \mathcal{H}^2$. Then
$$\int v(\Delta z)dx = \int(\Delta v)zdx \quad \text{ for all } z \in D$$
from which it follows that $\Delta v = w$, completing the Pr **oof**.

Theorem 2.12 *Let u be a weak solution of (1) for some $\lambda \in I\!\!R$ and $f \in L^2$. Then $u \in H^2$,*
$$-\Delta u = \lambda u + f \text{ almost everywhere on } I\!\!R^N$$
and
$$\min\left\{\frac{1}{N^2}, \lambda^2\right\}\|u\|_{H^2}^2 \leq |f|_2^2 + 2\lambda\left|\nabla u\right|_2^2.$$
In particular, for $\lambda < 0$,
$$G_\lambda : L^2 \to H^2 \text{ is a bounded linear operator}$$
with
$$\|G_\lambda f\|_{H^2} \leq \max\left\{N, \frac{1}{|\lambda|}\right\}|f|_2 \text{ for all } f \in L^2.$$

Proof From the definition of weak solution of (1) we know that

$$\int \nabla u.\nabla z dx = \int \{\lambda u + f\} z dx \text{ for all } z \in D.$$

Hence

$$\int u \Delta z dx = \int w z dx \text{ for all } z \in D$$

where $w = -\{\lambda u + f\} \in L^2$ and Lemma 2.11 yields $u \in H^2$ with $-\Delta u = \lambda u + f$ and

$$|\xi|^2 \widehat{u}(\xi) = \lambda \widehat{u}(\xi) + \widehat{f}(\xi) \text{ for almost all } \xi \in I\!\!R^N.$$

Furthermore, for $v \in \mathcal{S}$,

$$\begin{aligned} \left| \int (\partial_j \partial_k u) \overline{v} dx \right| &= \left| \int u \overline{(\partial_j \partial_k v)} dx \right| = \left| \int \widehat{u} \overline{(\widehat{\partial_j \partial_k v})} d\xi \right| \\ &= \left| \int \widehat{u} \xi_j \xi_k \overline{\widehat{v}} d\xi \right| \leq \int |\xi|^2 |\widehat{u}| |\widehat{v}| d\xi \\ &\leq \left| |\xi|^2 \widehat{u} \right|_2 |\widehat{v}|_2 \leq |\lambda u + f|_2 |v|_2 . \end{aligned}$$

From the density of $\mathcal{S}$ in $\mathcal{L}^2$ we conclude that

$$|\partial_j \partial_k u|_2 \leq |\lambda u + f|_2$$

and so

$$\begin{aligned} \sum_{j,k=1}^{N} |\partial_j \partial_k u|_2^2 &\leq N^2 |\lambda u + f|_2^2 \\ &= N^2 \left\{ \lambda^2 |u|_2^2 + |f|_2^2 + 2\lambda \int u f dx \right\}. \end{aligned}$$

But, putting $z = u$, we have

$$\int |\nabla u|^2 dx = \int \{\lambda u + f\} u dx$$

and so

$$\begin{aligned} \sum_{j,k=1}^{N} |\partial_j \partial_k u|_2^2 &\leq N^2 \left\{ \lambda^2 |u|_2^2 + |f|_2^2 + 2\lambda \int |\nabla u|^2 dx - 2\lambda^2 \int u^2 dx \right\} \\ &= N^2 \left\{ |f|_2^2 + 2\lambda |\nabla u|_2^2 - \lambda^2 |u|_2^2 \right\}. \end{aligned}$$

Hence

$$\begin{aligned}
|f|_2^2 + 2\lambda |\nabla u|_2^2 &\geq \lambda^2 |u|_2^2 + \frac{1}{N^2} \sum_{j,k=1}^{N} |\partial_j \partial_k u|_2^2 \\
&\geq \min\left\{\frac{1}{N^2}, \lambda^2\right\} \left\{ |u|_2^2 + \sum_{j,k=1}^{N} |\partial_j \partial_k u|_2^2 \right\} \\
&= \min\left\{\frac{1}{N^2}, \lambda^2\right\} \|u\|_{H^2}^2 .
\end{aligned}$$

2.5 $W^{2,p}$–regularity

In this section we obtain an optimal regularity result for solutions of the Helmholtz equation. It is based on the Calderon-Zygmund estimate which we state without proof. The results of this section will not be used in the subsequent discussion.

Theorem 2.13 *(Calderon-Zygmund) For $1 < p < N$, there is a constant $C(N,p)$ such that*

$$|\partial_i \partial_j u|_p \leq C(N,p) |\Delta u|_p \quad \textit{for all } u \in W^{2,p}$$

and all $i, j = 1, ..\ \ N$.

Proof See Gilbarg and Trudinger [6] Corollary 9.10.

Using this estimate we obtain the following result.

Theorem 2.14 *Let $\lambda < 0$ and $f \in L^p$ for some $p \in (1, \infty)$. Then there is a unique element $u \in W^{2,p}$ such that*

$$-\Delta u = \lambda u + f \qquad \textit{a.e. on } \mathbb{R}^N.$$

Proof Consider a sequence $\{f_n\} \subset D$ such that $|f - f_n|_p \to 0$. Let $u_n = T_k f_n$ be the solution of (11) defined by (10) where $k^2 = -\lambda$. By (14) and (16) we have that $u_n \in W^{1,p}$ and that $\{u_n\}$ is a Cauchy sequence in $W^{1,p}$. This implies that $\{\Delta u_n\}$is a Cauchy sequence in L^p since $\Delta u_n = -(\lambda u_n + f_n)$. The Calderon-Zygmund estimate now shows that $\{u_n\}$ is a Cauchy sequence in $W^{2,p}$. The limit of $\{u_n\}$ in $W^{2,p}$ satisfies $-\Delta u = \lambda u + f$ a.e. on $\mathbb{R}^N$.

To establish the uniqueness, suppose that $u \in W^{2,p}$ and that $-\Delta u = \lambda u$ a.e. on $\mathbb{R}^N$. Consider a sequence $\{u_n\} \subset D$ such that $|u - u_n|_{W^{2,p}} \to 0$. Then $\Delta u_n + \lambda u_n = r_n$ where $|r_n|_p \to 0$. Next, set $g(t) = |t|^{p-2}\, t$ for $t \in \mathbb{R}$ and let $\{g_m\}$ be a sequence of odd, continuously differentiable, increasing functions such that $g_m \to g$ uniformly on $\mathbb{R}$. Now,

$$\lambda \int |u|^p\, dx = \lambda \lim_{n\to\infty} \int |u_n|^p\, dx = \lambda \lim_{n\to\infty} \int u_n g(u_n) dx$$

and, for each n,

$$\begin{aligned}
\lambda \int u_n g(u_n) dx &= \lambda \lim_{m\to\infty} \int u_n g_m(u_n) dx \\
&= \lim_{m\to\infty} \int (-\Delta u_n - r_n) g_m(u_n) dx \\
&= \lim_{m\to\infty} \int |\nabla u_n|^2\, g_m'(u_n) dx - \int r_n g(u_n) dx \\
&\geq -\int r_n g(u_n) dx \text{ since } g_m \text{ is increasing.}
\end{aligned}$$

Hence

$$\lambda \int |u|^p\, dx \geq -\lim_{n\to\infty} \int r_n g(u_n) dx.$$

But

$$\left| \int r_n g(u_n) dx \right| \leq |r_n|_p\, |g(u_n)|_q \text{ where } \frac{1}{p} + \frac{1}{q} = 1$$

and

$$|g(u_n)|_q^q = \int |u_n|^{(p-1)q}\, dx = |u_n|_p^p.$$

Since $|r_n|_p \to 0$ and $|u_n|_p^p \to |u|_p^p$, it follows that $\lim_{n\to\infty} \int r_n g(u_n) dx = 0$. Hence $\lambda \int |u|^p\, dx \geq 0$ where $\lambda < 0$. This shows that $u \equiv 0$ and the uniqueness of the solution in $W^{2,p}$ follows immediately.

Corollary 2.15 *Let $f \in L^p \cap L^q$ where $p \in (\frac{2N}{N+2}, 2]$ and $q \geq p$. Let u be a weak solution of (1) for f and some $\lambda \in \mathbb{R}$. Then $u \in W^{2,q}$.*

Proof By Theorem 2.8, $u \in L^p \cap L^q$ and so $F \in L^p \cap L^q$ where $F = (\lambda+1)u + f$. Let $v \in W^{2,q}$ be the unique solution of the equation $-\Delta v = -v + F$ in $W^{2,q}$ given by Theorem 2.14. Since $q \geq p > \frac{2N}{N+2}$, it follows from the Sobolev imbeddings that $v \in H^1$. Thus v is a weak solution of $-\Delta v = -v + F$ where $F \in L^p$. But u is also a weak solution of this equation and so by the uniqueness statement in Theorem 2.3, we must have that $u = v \in W^{2,q}$ as required.

2.6 Regularity for nonlinear equations

The above results can be applied to establish the regularity of weak solutions of more general, even nonlinear, equations. To exhibit this let us consider,

$$\begin{aligned} -\Delta u(x) + V(x)u(x) + W(x)\,|u(x)|^{\sigma}\,u(x) &= 0 \\ u &\in H^1 \end{aligned} \tag{19}$$

where $V, W \in L^{\infty}$ and $1 \leq \sigma + 1 < 2^*$ where 2^* is the critical exponent defined by

$$\begin{aligned} 2^* &= \infty \quad \text{for } N = 2 \\ 2^* &= \frac{N+2}{N-2} \quad \text{for } N \geq 3. \end{aligned}$$

Noting that $f \equiv Vu + W\,|u|^{\sigma}\,u \in L^1_{loc}$ whenever $u \in H^1$, we can define a weak solution of (19) as follows.

Definition 2.16 *Given $V, W \in L^{\infty}$ and $1 \leq \sigma + 1 < 2^*, u$ is a weak solution of (19) $\Longleftrightarrow u \in H^1$ and*

$$\int \nabla u.\nabla v dx = -\int \{Vu + W\,|u|^{\sigma}\,u\}\, v dx \text{ for all } v \in D.$$

We now show that all weak solutions are smooth and decay to zero at infinity.

Theorem 2.17 *Let $V, W \in L^{\infty}$ and $1 \leq \sigma + 1 < 2^*$ and suppose that u is a weak solution of (19).*

(i) Then $u \in W^{1,s}$ for all $2 \leq s \leq \infty$. In particular, $u \in C$ and $\lim_{|x| \to \infty} u(x) = 0$.

(ii) If, in addition, $u \geq 0$ on $\mathbb{R}^N$, then either $u \equiv 0$ or $u(x) > 0$ for all $x \in \mathbb{R}^N$.

Proof Set $f \equiv -(Vu + W\,|u|^{\sigma}\,u)$. It follows that for all $u \in H^1$,

$$f \in L^p \text{ where } \begin{cases} 2 \leq (\sigma+1)p < \infty \text{ if } N = 2 \\ 2 \leq (\sigma+1)p \leq \frac{2N}{N-2} \text{ if } N \geq 3. \end{cases}$$

Now

$$\left[\frac{2}{\sigma+1}, \frac{2N}{N-2}\,\frac{1}{(\sigma+1)}\right] \cap \left(\frac{2N}{N+2}, 2\right] \neq \emptyset$$

since the restrictions on σ ensure that $\frac{2}{\sigma+1} \leq 2$ and $\frac{2N}{N-2}\,\frac{1}{(\sigma+1)} > \frac{2N}{N+2}$.
Thus u is a weak solution of (1) for f and $\lambda = 0$ where $f \in L^p$ for some $p \in \left(\frac{2N}{N+2}, 2\right]$. From Theorem 2.8, $u \in W^{1,s}$ for some $s > 2$ and so

$$f \in L^p \text{ where } \begin{cases} 2 \leq (\sigma+1)p \leq \infty \text{ if } s > N \\ 2 \leq (\sigma+1)p < \infty \text{ if } s = N \\ 2 \leq (\sigma+1)p \leq \frac{Ns}{N-s} \text{ if } s < N. \end{cases}$$

Noting that $\frac{Ns}{N-s} > \frac{2N}{N-2}$ for $2 < s < N$, we see that a bootstrap procedure establishes part (i).

(ii) By part (i) we know that $u \in L^\infty$ and so there exists $k > 0$ such that

$$k^2 - V - W\,|u|^\sigma > 0 \text{ on } \mathbb{R}^N.$$

Setting $g = \{k^2 - V - W\,|u|^\sigma\}\,u$, we see that $g \geq 0$ on $\mathbb{R}^N$ and that $g \in L^s$ for $2 \leq s \leq \infty$ by (i). By Theorem 2.9 this implies that $u(x) > 0$ on $\mathbb{R}^N$ unless $g \equiv 0$. But $g \equiv 0 \Longleftrightarrow u \equiv 0$.

3 The Schrödinger equation

In this part we discuss the Schrödinger equation with a bounded potential V. The first step is to show that $u \longmapsto -\Delta u + Vu$ can be considered as an unbounded self-adjoint operator acting in L^2. This is done in the first section, once the basic notions from functional analysis concerning self-adjoint operators and their spectra have been recalled. The next section contains the main results concerning the spectrum of the Schrödinger equation. Since the Helmholtz equation corresponds to the case $V \equiv 0$, these results complete the discussion of this equation. In the final section we establish some of the important properties of eigenfunctions of the Schrödinger operator.

3.1 Self-adjointness

We begin by recalling the basic definitions concerning self-adjoint operators, Weidmann [1] or Berezin and Shubin [4].

Let $(H, \langle\cdot,\cdot\rangle)$ be a real Hilbert space.

Definition 3.1 *Let $L : D(L) \subset H \to H$ be a linear operator whose domain $D(L)$ is a dense subspace of H. Its adjoint operator, $L^* : D(L^*) \subset H \to H$ is defined as follows:*

$$v \in D(L^*) \Longleftrightarrow \begin{cases} v \in H \text{ and there exists an element } w \in H \\ \text{such that } \langle Lu, v\rangle = \langle u, w\rangle \text{ for all } u \in D(L) \end{cases}$$

and

$$L^*v = w \text{ for all } v \in D(L^*)$$

where w is the (unique, by the density of $D(L)$ in H) element associated with v in the definition of $D(L^*)$.

The operator $L : D(L) \subset H \to H$ is said to be self-adjoint $\Longleftrightarrow L = L^*$ in the sense that $D(L) = D(L^*)$ and $L^*v = Lv$ for all $v \in D(L^*)$.

Note that $0 \in D(L^*)$ which is a subspace of H. It is also easily seen that $L^* : D(L^*) \subset H \to H$ is always a closed linear operator and that $\ker(L^*) = R(L)^\perp$, Weidmann [1] Theorems 4.13 and5.3. Hence a self-adjoint operator is always closed. Before recalling the notion of spectrum of a self-adjoint operator, it is worth noting the following equivalence between an algebraic and metric characterization of regular value.

Lemma 3.2 *Let $L : D(L) \subset H \to H$ be a self-adjoint operator in a real Hilbert space $(H, \langle \cdot, \cdot \rangle)$. For $\lambda \in I\!R$, we have that*

$$L - \lambda I : D(L) \to H \text{ is an isomorphism}$$

$\Longleftrightarrow$

$$\exists\ c > 0 \text{ such that } \|(L - \lambda I)u\| \geq c\,\|u\| \text{ for all } u \in D(L).$$

Proof If $L - \lambda I : D(L) \to H$ is an isomorphism, $(L - \lambda I)^{-1}$ is defined on H and is a closed operator since $L : D(L) \subset H \to H$ is closed and the inverse of a closed operator is always closed, Weidmann [1] Proposition on page 89. The closed graph theorem, Weidmann [1] Theorem 5.6, shows that $(L - \lambda I)^{-1} : H \to H$ is bounded and so there exists a constant $M > 0$ such that

$$\left\|(L - \lambda I)^{-1} v\right\| \leq M\,\|v\| \text{ for all } v \in H.$$

Setting $v = (L - \lambda L)u$ for $u \in D(L)$, this means that

$$\|(L - \lambda I)u\| \geq \frac{1}{M}\,\|u\|\,.$$

Conversely, if

$$\exists\ c > 0 \text{ such that } \|(L - \lambda I)u\| \geq c\,\|u\| \text{ for all } u \in D(L),$$

it follows that $L - \lambda I : D(L) \subset H \to H$ is injective and that

$$(L - \lambda I)^{-1} : R(L - \lambda I) \to D(L - \lambda I) = D(L) \text{ is bounded.}$$

But, as we have already noted, this operator is also closed. Being closed and bounded its domain, $R(L - \lambda I)$, must be a closed subspace of H, Weidmann [1] Theorem 5.6. Now, as we already observed,

$$[R(L - \lambda I)]^{\perp} = \ker(L - \lambda I)^*$$

where

$$\ker(L - \lambda I)^* = \ker(L - \lambda I) = \{0\}$$

by the self-adjointness and injectivity of $L - \lambda I$. Thus $R(L - \lambda I) = H$ since it is dense and closed, proving that $L - \lambda I : D(L) \subset H \to H$ is an isomorphism.

For self-adjoint operators the basic notions concerning the spectrum are now recalled.

Definition 3.3 *Let $L : D(L) \subset H \to H$ be a self-adjoint operator. Its resolvent set is*

$$\rho(L) = \{\lambda \in I\!\!R : L - \lambda I : D(L) \to H \text{ is an isomorphism}\}$$

and its spectrum is the set

$$\sigma(L) = I\!\!R \backslash \rho(L).$$

The elements of $\rho(L)$ are called regular values for $L : D(L) \subset H \to H$. The following subsets of $\sigma(L)$ are of primary importance. The point spectrum is the set

$$\sigma_p(L) = \{\lambda \in I\!\!R : \ker(L - \lambda I) \neq \{0\}\},$$

its elements being the eigenvalues of L. The discrete spectrum is the set

$$\sigma_d(L) = \{\lambda \in \sigma_p(L) : \dim\ker(L - \lambda I) < \infty \text{ and } \lambda \text{ is an isolated point of } \sigma(L)\}$$

and its complement in $\sigma(L)$ is called the essential spectrum

$$\sigma_e(L) = \sigma(L) \backslash \sigma_d(L).$$

We note that $\sigma(L)$ and $\sigma_e(L)$ are closed subsets of $I\!\!R$ for every self-adjoint operator L. The elements of $\sigma_d(L)$ are the isolated eigenvalues of finite multiplicity of L. There are several reasons for preferring the decomposition of $\sigma(L)$ through $\sigma_d(L)$ and its complement rather than that based on $\sigma_p(L)$. Let us mention some of them.

$$\sigma_d(L) =$$
$$\{\lambda \in \sigma(L) : L - \lambda I : D(L) \to H \text{ is a Fredholm operator }\},$$

and

$$\sigma_e(L) = \sigma_e(L + K)$$

for every compact self-adjoint operator $K : H \to H$. In what follows we shall not need these two properties so we do not include their proofs. On the other hand the following elementary result will be used several times.

Lemma 3.4 *Let $L : H \to H$ be a bounded self-adjoint operator and set*

$$\begin{aligned} m &= \inf \{\langle Lu, u\rangle : u \in H \textit{ and } \|u\| = 1\}, \\ M &= \sup \{\langle Lu, u\rangle : u \in H \textit{ and } \|u\| = 1\}. \end{aligned}$$

Then

$$\begin{aligned} (i) \quad \sigma(L) &\subset [m, M], \\ (ii) \quad \|L\| &= \sup\{|\lambda| : \lambda \in \sigma(L)\} = \max\{|m|, |M|\} \qquad \textit{and} \\ (iii) \quad m, M &\in \sigma(L). \end{aligned}$$

Proof (i) If $\lambda > M$, for all $u \in H$,

$$\begin{aligned} \langle (L - \lambda I)u, u\rangle &\leq (M - \lambda)\|u\|^2 \qquad \text{and so} \\ \|(L - \lambda I)u\| \, \|u\| &\geq |\langle (L - \lambda I)u, u\rangle| \\ &\geq -\langle (L - \lambda I)u, u\rangle \geq (\lambda - M)\|u\|^2. \end{aligned}$$

Thus

$$\|(L - \lambda I)u\| \geq (\lambda - M)\|u\|$$

and $\lambda \in \rho(L)$ by Lemma 3.2. Hence we see that $\sigma(L) \subset (-\infty, M]$ and a similar argument shows that $\sigma(L) \subset [m, \infty)$.

(ii) Let $\mu = \max\{|m|, |M|\}$. Clearly $\mu = \sup\{|\langle Lu, u\rangle| : u \in H \text{ and } \|u\| = 1\}$ so it is enough to show that $\|L\| = \mu$.

For all $u \in H$,

$$|\langle Lu, u\rangle| \leq \|Lu\| \, \|u\| \leq \|L\| \, \|u\|^2$$

and so $\mu \leq \|L\|$. Conversely, for all $u, v \in H$,

$$\begin{aligned} \langle Lu, v\rangle &= \frac{1}{4}\{\langle L(u+v), u+v\rangle - \langle L(u-v), u-v\rangle\} \\ &\leq \frac{\mu}{4}\left\{\|u+v\|^2 + \|u-v\|^2\right\} = \frac{\mu}{2}\left\{\|u\|^2 + \|v\|^2\right\} \\ &= \mu \quad \text{if } \|u\| = \|v\| = 1. \end{aligned}$$

For $u \in H$ with $Lu \neq 0$ we set $v = Lu/\|Lu\|$ and obtain

$$\|Lu\| \leq \mu \quad \text{for all } u \in H \text{ with } \|u\| = 1.$$

Thus $\|L\| \leq \mu$.

(iii) If $m = M$, it follows from (ii), applied to $L - MI$ that $\|(L - MI)\| = 0$. Hence $L = MI$ and $\sigma(L) = \{M\}$ if $m = M$.

If $m < M$, then, by replacing L by $L - mI$, we can suppose that $M > 0 = m$. To prove that $M \in \sigma(L)$ it is enough, by Lemma 3.2, to show that there exists a sequence $\{u_n\} \subset H$ such that $\|u_n\| = 1$ for all $n \in N$ and $\|(L - MI)u_n\| \to 0$ as $n \to \infty$. But there exists a sequence $\{u_n\} \subset H$ such that $\|u_n\| = 1$ for all $n \in N$ and $0 \leq M - \langle Lu_n, u_n \rangle \leq \frac{1}{n}$. Thus

$$\begin{aligned} \|(L - MI)u_n\|^2 &= \|Lu_n\|^2 + M^2 - 2M \langle Lu_n, u_n \rangle \\ &\leq 2M^2 + 2M \left\{ \frac{1}{n} - M \right\} \text{ by (ii).} \end{aligned}$$

Thus $M \in \sigma(L)$ and a similar argument shows that $m \in \sigma(L)$.

We are now ready to begin the study of the Schrödinger operator which we define as follows.

Definition 3.5 *Given $V \in L^\infty$, we define the Schrödinger operator $S : D(S) \subset L^2 \to L^2$ generated by the potential V by*

$$D(S) = H^2 \text{ and } Su = -\Delta u + Vu \text{ for } u \in H^2.$$

This operator is self-adjoint.

Theorem 3.6 *For $V \in L^\infty$, the Schrödinger operator $S : D(S) \subset L^2 \to L^2$ generated by the potential V is self-adjoint.*

Proof Note that H^2 is dense in L^2 so the adjoint operator $S^* : D(S^*) \subset L^2 \to L^2$ is well-defined. Furthermore, for all $u, v \in H^2$,

$$\int (Su)v dx = \int (-\Delta u + Vu)v dx = \int u(-\Delta v + Vv) dx$$

where $-\Delta v + Vv \in L^2$. This already shows that $H^2 \subset D(S^*)$ and that $S^* v = -\Delta v + Vv = Sv$ for all $v \in H^2$.

On the other hand if $v \in D(S^*)$, then $v \in L^2$ and there exists an element $w \in L^2$ such that

$$\int (Su)v dx = \int uw dx \text{ for all } u \in D(S) = H^2.$$

Thus

$$\int(-\Delta u + Vu)v dx = \int uw dx \text{ for all } u \in D$$

and so

$$\int v(\Delta u) dx = \int (Vv - w)u dx \text{ for all } u \in D$$

where v and $(Vv - w) \in L^2$.
Referring to Lemma 2.11, we see that this implies that $v \in H^2$ and that $\Delta v = Vv - w$. This shows that $D(S^*) \subset H^2$, completing the proof.

Corollary 3.7 *Let $V \in L^\infty$. Then*

$$\lambda \in \rho(S) \Longleftrightarrow \begin{cases} (S-\lambda I) : H^2 \to L^2 \textit{ is injective and} \\ (S-\lambda I)^{-1} : R(S-\lambda I) \subset L^2 \to H^2 \textit{ is bounded.} \end{cases}$$

Thus, for $\lambda \in \rho(S), (S - \lambda I) : H^2 \to L^2$ is both an isomorphism and a homeomorphism. (In these statements H^2 is considered with its Sobolev norm.)

Proof If $\lambda \in \rho(S), (S - \lambda I) : H^2 \to L^2$ is bijective and continuous. The Closed Graph Theorem, Weidmann [1] Theorem 5.6, shows that $(S - \lambda I)^{-1} : L^2 \to H^2$ is continuous as required.

Conversely, if $(S-\lambda I) : H^2 \to L^2$ is injective and $(S-\lambda I)^{-1} : R(S-\lambda I) \subset L^2 \to H^2$ is bounded, we certainly have that $(S - \lambda I)^{-1} : R(S - \lambda I) \subset L^2 \to L^2$ is bounded and so by Lemma 3.2, $\lambda \in \rho(S)$.

3.2 The spectrum of S

One of the main features of the spectral theory of elliptic equations on $\mathbb{R}^N$, as opposed to their study on bounded domains, is the fact that the spectrum contains points which are not eigenvalues. We begin the description of the spectrum of S by showing a striking example of this.

Theorem 3.8 *For $V \equiv 0$,*

$$\sigma_p(S) = \emptyset.$$

Proof Suppose that $u \in \ker(S - \lambda I)$ for some $\lambda \in \mathbb{R}$. Then

$$u \in H^2 \text{ and } -\Delta u = \lambda u,$$

Hence we have that $u \in L^2$ and

$$\int u \Delta z dx = \int (-\lambda u) z dx \text{ for all } z \in D.$$

It follows from Lemma 2.11 that

$$-|\xi|^2 \widehat{u}(\xi) = -\lambda \widehat{u}(\xi) \text{ for almost all } \xi \in I\!R^N.$$

Since $\left\{\xi \in I\!R^N : |\xi|^2 = \lambda\right\}$ has $N-$dimensional zero, this implies that $\widehat{u}(\xi) = 0$ for almost all $\xi \in I\!R^N$. Thus $u \equiv 0$ and $\ker(S - \lambda I) = \{0\}$.

Remark 3.1 *This result shows that for $V \equiv 0$, $\sigma(S) = \sigma_e(S)$ since $\sigma_d(S) \subset \sigma_p(S)$. We shall soon see that for any $V \in L^\infty, \sigma_e(S) \neq \emptyset$.*

The following quantity plays a fundamental role since it characterizes the infimum of the spectrum of S. For any $V \in L^\infty$, we set

$$\Lambda = \inf\left\{\int |\nabla u|^2 + V u^2 dx : u \in H^1 \text{ and } \int u^2 dx = 1\right\}. \tag{20}$$

Lemma 3.9 *Let $V \in L^\infty$. Then*

(1) $\Lambda \geq -|V|_\infty > -\infty$.

(2)
$$\Lambda = \inf\left\{\int |\nabla u|^2 + V u^2 dx : u \in D \text{ and } \int u^2 dx = 1\right\}$$

and so we also have,

$$\Lambda = \inf\left\{\int (Su) u dx : u \in H^2 \text{ and } \int u^2 dx = 1\right\}.$$

(3) If $u \in H^1$ with $\int u^2 dx = 1$ and $\int |\nabla u|^2 + V u^2 dx = \Lambda$, then

$$u \in H^2, u \in \ker(S - \Lambda I) \text{ and } \Lambda \in \sigma_p(S).$$

Proof (i) For $u \in H^1$,

$$\int |\nabla u|^2 + V u^2 dx \geq \int V u^2 dx \geq -|V|_\infty \int u^2 dx.$$

(ii) Since $D \subset H^1$, we clearly have that

$$\Lambda \leq \inf \left\{ \int |\nabla u|^2 + V u^2 dx : u \in D \text{ and } \int u^2 dx = 1 \right\}.$$

Setting

$$Q(u) = \int |\nabla u|^2 + V u^2 dx,$$

we easily see that $Q \in C^1(H^1)$.
There is a sequence $\{u_n\} \subset H^1$ such that $|u_n|_2 = 1$ and $Q(u_n) \to \Lambda$. Furthermore, for all $n \in N$, there is an element $v_n \in D$ such that $\|v_n - u_n\|_{H^1} \leq \frac{1}{n}$. Hence,

$$|v_n|_2^2 = \int (u_n^2 + v_n^2 - u_n^2) dx = 1 + \int (v_n - u_n)(v_n + u_n) dx$$

and so

$$\left| |v_n|_2^2 - 1 \right| \leq |v_n - u_n|_2 \, |v_n + u_n|_2 \leq \frac{1}{n} \{ |v_n - u_n|_2 + 2 \, |u_n|_2 \} \leq \frac{1}{n} \left\{ 2 + \frac{1}{n} \right\}.$$

Thus, $|v_n|_2 \to 1$ and we set

$$w_n = \frac{v_n}{|v_n|_2}.$$

Then, $Q(w_n) = \frac{Q(v_n)}{|v_n|_2}$ and

$$Q(v_n) - Q(u_n) = \int |\nabla v_n|^2 - |\nabla u_n|^2 + V \left\{ v_n^2 - u_n^2 \right\} dx,$$

so that

$$\begin{aligned}
& |Q(v_n) - \Lambda| \leq |Q(v_n) - Q(u_n)| + |Q(u_n) - \Lambda| \\
\leq \; & |\nabla v_n - \nabla u_n|_2 \, |\nabla v_n + \nabla u_n|_2 + |V|_\infty \, |v_n - u_n|_2 \, |v_n + u_n|_2 + |Q(u_n) - \Lambda| \\
\leq \; & \frac{1}{n} \left\{ \frac{1}{n} + 2 \, |\nabla u_n|_2 \right\} + |V|_\infty \, \frac{1}{n} \left\{ \frac{1}{n} + 2 \right\} + |Q(u_n) - \Lambda| \, .
\end{aligned}$$

But there is an n_0 such that $Q(u_n) \leq \Lambda + 1$ for all $n \geq n_0$ and hence,

$$\int |\nabla u_n|^2 \, dx \leq \Lambda + 1 - \int V u_n^2 dx \leq \Lambda + 1 + |V|_\infty \text{ for all } n \geq n_0.$$

It now follows that $Q(v_n) \to \Lambda$. Since $|v_n|_2 \to 1$ we also have that $Q(w_n) \to \Lambda$, showing that

$$\Lambda \geq \inf \left\{ \int |\nabla u|^2 + Vu^2 dx : u \in D \text{ and } \int u^2 dx = 1 \right\} \geq \Lambda.$$

(iii) Setting $P(u) = \int u^2 dx$, we have that $P \in C^1(H^1)$ and

$$P'(u)v = 2 \int uvdx \text{ for all } u, v \in H^1.$$

In particular, $P'(u)u = 2|u|_2^2 \neq 0$ for $u \neq 0$. Thus, if $u \in H^1$ with $P(u) = 1$ and $Q(u) = \Lambda$, there is a Lagrange multiplier $\xi \in I\!R$ (see Theorem 26.1 of Deimling [9], for example) such that

$$Q'(u)v = \xi P'(u)v \text{ for all } v \in H^1.$$

That is,

$$2 \int (\nabla u . \nabla v + Vuv) dx = \xi 2 \int uvdx \text{ for all } v \in H^1$$

and, putting $v = u$, we find that $2Q(u) = 2\xi$. Hence $\xi = Q(u) = \Lambda$ and consequently

$$\int (\nabla u . \nabla v) dx = \int (\Lambda - V) uvdx \text{ for all } v \in H^1.$$

It now follows from Theorem 2.12 (with $\lambda = \Lambda$ and $f = -Vu$) that $u \in H^2$ and $-\Delta u = \Lambda u - Vu$. Thus $u \in D(S)$ and $Su = \lambda u$.

Remark 3.2 *Comparing part (iii) of the above lemma with the Theorem 3.8 we see that when $V \equiv 0$ the value $\Lambda = \inf \left\{ \int |\nabla u|^2 dx : u \in H^1 \text{ and } \int u^2 dx = 1 \right\}$ is not attained by any element $u \in H^1$ with $\int u^2 dx = 1$. However as the next result shows, for any $V \in L^\infty$,*

$$\Lambda \in \sigma(S) \text{ and } \Lambda = \inf \sigma(S).$$

Theorem 3.10 *Let $V \in L^\infty$. Then,*

$$(i) \quad \sigma(S) \subset [\Lambda, \infty) \quad \text{and} \quad (ii) \quad \Lambda \in \sigma(S)$$

where Λ is defined by (20).

Proof (i) By Lemma 3.9(2), for all $u \in H^2$,

$$\Lambda \int u^2 dx \leq \int (Su) u dx$$

and so, for all $\lambda \in I\!R$,

$$(\Lambda - \lambda) |u|_2^2 \leq \int \{(S - \lambda I)u\} u dx \leq |(S - \lambda I)u|_2 |u|_2 .$$

Thus

$$|(S - \lambda I)u|_2 \geq (\Lambda - \lambda) |u|_2 \text{ for all } u \in D(S)$$

and it follows from Lemma 3.2 that $\lambda \in \rho(S)$ if $\Lambda - \lambda > 0$.

(ii) By part (i) we know that $\sigma(S) \subset [\Lambda, \infty)$. Let $m \geq \Lambda$ be such that $\sigma(S) \subset [m, \infty)$. To complete the Pr**oof** we shall show that $m \leq \Lambda$. We choose any $\xi \in (-\infty, m)$. Since $\xi \in \rho(S)$, we can set

$$A = (S - \xi I)^{-1}$$

and we have that

$$A : L^2 \to L^2 \text{ is a bounded self-adjoint linear operator.}$$

Furthermore, $0 \in \sigma(A)$ since $R(A) = D(S) = H^2 \neq L^2$. For $\lambda \neq 0$,

$$A - \lambda I = \lambda \left\{ \frac{1}{\lambda} I - (S - \xi I) \right\} A = \lambda \left\{ (\frac{1}{\lambda} + \xi) I - S \right\} A$$

and so

$$\begin{aligned} A - \lambda I \quad &: \quad L^2 \to L^2 \text{ is an isomorphism} \\ &\iff \quad S - (\frac{1}{\lambda} + \xi) I : H^2 \to L^2 \\ &\iff \quad (\frac{1}{\lambda} + \xi) \in \rho(S). \end{aligned}$$

Thus we see that

$$\sigma(A) = \{0\} \cup \left\{ \frac{1}{\mu - \xi} : \mu \in \sigma(S) \right\}$$

and consequently

$$\sigma(A) \subset \left[0, \frac{1}{m-\xi}\right].$$

By Lemma 3.4, this implies that

$$\int (Av)v dx \geq 0 \quad \text{for all } v \in L^2.$$

For any $u \in H^2$,we set $v = (S - \xi I)u$ and obtain

$$\int [(S - \xi I)u]\, u dx = \int (Av)v dx \geq 0.$$

This shows that $\int (Su)u dx \geq \xi \int u^2 dx$ for all $u \in H^2$ and it follows from Lemma 3.9(2) that $\xi \leq \Lambda$. But ξ is an arbitrary number less than m. We can therefore conclude that $m \leq \Lambda$, completing the proof.

This result shows that the spectrum of S is never empty and characterizes its infimum. We shall now obtain similar information about the essential spectrum. The fact that the essential spectrum is non-empty for any $V \in L^\infty$ is a simple consequence of the non-compactness of the embedding of H^2 in L^2.

Theorem 3.11 *Let $V \in L^\infty$. Then*

$$(i) \text{ for all } \lambda \in \rho(S), (S - \lambda I)^{-1} : L^2 \to L^2 \text{ is not compact}$$

and

$$(ii) \quad \sigma_e(S) \neq \emptyset.$$

Proof (i) Let $\varphi \in D$ with $\varphi \not\equiv 0$. For $n \in N$, set $\varphi_n(x) = \varphi(x-n)$ for $x \in \mathbb{R}^N$. Then $\varphi_n \in H^2$ and $\|\varphi_n\|_{H^2} = \|\varphi\|_{H^2} > 0$ for all $n \in N$. Furthermore,

$$\int \varphi_n v dx \to 0 \text{ as } n \to \infty \quad \text{for all } v \in D$$

and so, since $\{\varphi_n\}$ is bounded in L^2, it follows, from Weidmann [1] Theorem 4.24(b), that $\varphi_n \rightharpoonup 0$ weakly in L^2. This means that if $\{\varphi_n\}$ had a subsequence $\{\varphi_{n_i}\}$ converging strongly in L^2 its limit would have to be 0. But then we would have

$$0 = \lim_{i\to\infty} |\varphi_{n_i}|_2 = |\varphi|_2 > 0.$$

Hence we see that $\{\varphi_n\}$ cannot have a subsequence converging strongly in L^2.

Let $\lambda \in \rho(S)$ and set $T = (S - \lambda I)^{-1}$. We consider the sequence $\{v_n\}$ where $v_n = (S - \lambda I)\varphi_n$. It is bounded in L^2 since

$$|v_n|_2 = |(-\Delta + V - \lambda)\varphi_n|_2 \leq (N + |V|_\infty + |\lambda|) \|\varphi_n\|_{H^2}.$$

But $Tv_n = \varphi_n$ and so $\{v_n\}$ is a bounded sequence in L^2 for which $\{Tv_n\}$ has no convergent subsequence. This means that $T : L^2 \to L^2$ is not compact.

(ii) Suppose that $\sigma_e(S) = \emptyset$. Then there is a sequence $\{\mu_n : n \in N\}$ such that

$$\begin{aligned} \{\mu_n\} &= \sigma(S) \subset [\Lambda, \infty), \quad 0 < \dim\ker(S - \mu_n I) < \infty \quad \text{for all } n \in N \\ &\text{and} \\ \text{for every } R &\in (\Lambda, \infty), \{\mu_n\} \cap [\Lambda, R] \text{ contains only a finite number of points.} \end{aligned}$$

Choosing $\xi < \Lambda$ and setting $A = (S - \xi I)^{-1}$, we recall from the Pr **oof** of the preceding theorem that

$$A : L^2 \to L^2 \text{ is a bounded self-adjoint linear operator}$$

and that

$$\sigma(A) = \{0\} \cup \left\{ \frac{1}{\mu - \xi} : \mu \in \sigma(S) \right\} = \{0\} \cup \left\{ \frac{1}{\mu_n - \xi} : n \in N \right\}.$$

Hence, for every $\varepsilon > 0$, $\sigma(A)\backslash[-\varepsilon, \varepsilon]$ contains only a finite number of points and each of them is an eigenvalue of finite multiplicity. Let E_ε be the subspace of L^2 spanned by the all the eigenvectors of A associated with eigenvalues in $\sigma(A)\backslash[-\varepsilon, \varepsilon]$. Then $\dim E_\varepsilon < \infty$ and $A(E_\varepsilon) = E_\varepsilon$. Furthermore the self-adjointness of A implies that $A(E_\varepsilon^\perp) \subset E_\varepsilon^\perp$ and that $\sigma(A_\varepsilon) = \sigma(A) \cap [-\varepsilon, \varepsilon]$ where $A_\varepsilon : E_\varepsilon^\perp \to E_\varepsilon^\perp$ denotes the restriction of A to $E_\varepsilon^\perp$. It follows from Lemma 3.4 that $\|A_\varepsilon\| \leq \varepsilon$.

Using $P_\varepsilon : L^2 \to L^2$ to denote the orthogonal projection of L^2 onto E_ε, we can write

$$A = AP_\varepsilon + A(I - P_\varepsilon)$$

where $AP_\varepsilon : L^2 \to L^2$ is compact and $\|A_\varepsilon(I - P_\varepsilon)\| = \|A_\varepsilon\| \leq \varepsilon$. Thus we see that there is a sequence $\left\{AP_{\frac{1}{n}}\right\}$ of compact linear operators such that

$\left\|A - AP_{\frac{1}{n}}\right\| = \left\|A(I - P_{\frac{1}{n}})\right\| = \left\|A_{\frac{1}{n}}\right\| \leq \frac{1}{n}$. Since the compact operators form a closed subspace of the space of all bounded linear operators on L^2, this proves that $A : L^2 \to L^2$ is compact, contradicting part (i). Hence $\sigma_e(S) \neq \emptyset$.

Just as Theorem 3.10 provides a characterization of the infimum of the spectrum of S through the quantity Λ defined by (20), we can also provide a lower bound for the infimum of the essential spectrum of S which, under some additional assumptions, is sharp. As we show in Theorem 3.15 below, this is given by a more explicit quantity than Λ, namely,

$$l \equiv \lim_{R\to\infty} ess \inf_{|x|\geq R} V(x). \tag{21}$$

In what follows we shall sometimes express the definition of l more concisely as

$$l \equiv \lim \inf_{|x|\to\infty} V(x).$$

It is convenient to establish some preliminary results which are of independent interest.

Lemma 3.12 *For $V \in L^\infty$ we define an operator $M : L^2 \to L^2$ by $Mu(x) = V(x)u(x)$ for $x \in I\!R^N$.*

(i) $M : L^2 \to L^2$ is a bounded self-adjoint operator.

(ii) If

$$\lim_{R\to\infty} ess \sup_{|x|\geq R} |V(x)| = 0, \qquad \textit{then } M : H^1 \to L^2 \textit{ is compact.}$$

Proof (i) For $u, v \in L^2$,

$$|Mu|_2^2 = \int (Vu)^2 dx \leq |V|_\infty^2 \, |u|_2^2$$

and

$$\int (Mu)v dx = \int Vuv dx = \int u(Mv) dx.$$

(ii) For $\rho > 0$, let

$$\chi_\rho(x) = \begin{cases} 1 & \text{for } |x| \leq \rho \\ 0 & \text{for } |x| > \rho \end{cases}.$$

Since $H^1(B(0,\rho))$ is compactly embedded in $L^2(B(0,\rho))$, it follows easily that $u \longmapsto \chi_\rho V u$ is a compact linear operator from $H^1(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$. But, for $u \in H^1(\mathbb{R}^N)$,

$$|Mu - \chi_\rho Vu|_2^2 = \int_{|x|>\rho} (Vu)^2 dx \leq \sup_{|x|>\rho} |V(x)|^2 |u|_2^2 \leq \sup_{|x|>\rho} |V(x)|^2 \|u\|_{H^1}^2 .$$

Letting $\rho \to \infty$, we see that $M : H^1 \to L^2$ is the limit in $B(H^1, L^2)$ of a sequence of compact linear operators and so is itself compact, Weidmann [1] Theorem 6.4(e).

Lemma 3.13 *Let $V \in L^\infty$. Then $R(S - \lambda I)$ is a closed subspace of L^2 for all $\lambda < l$.*

Proof For any $u \in H^2$ and $\lambda \in \mathbb{R}$, we have that

$$(S - \lambda I)u = -\Delta u + (V - l)^+ u - (V - l)^- u - (\lambda - l)u.$$

Setting $Tu = -\Delta u + (V - l)^+ u$ for $u \in H^2$, we see that $T : H^2 \subset L^2 \to L^2$ is self-adjoint by Theorem 3.6 and then, by Theorem 3.10, that $\sigma(T) \subset [0, \infty)$ since $(V - l)^+ \geq 0$ on $\mathbb{R}^N$. Thus $T - (\lambda - l)I : H^2 \to L^2$ is an isomorphism for $\lambda < l$.

Furthermore, since $\lim_{|x|\to\infty}(V(x) - l)^- = 0$, it follows from Lemma 3.12 that the operator M, defined by $Mu = (V(x) - l)^- u$, is a compact linear operator from H^1 (and so "a fortiori" from H^2) into L^2. Hence for $\lambda < l$,

$$S - \lambda I = (T - (\lambda - l)I) \left\{ I - (T - (\lambda - l)I)^{-1} M \right\}$$

where $K = (T - (\lambda - l)I)^{-1} M : H^2 \to H^2$ is a compact linear operator because $(T - (\lambda - l)I)^{-1} : L^2 \to H^2$ is bounded by Corollary 3.7. Thus, by Weidmann [1] Theorem 6.6, $(I - K)H^2$ is a closed subspace of H^2 and consequently $R(S - \lambda I) = (T - (\lambda - l)I)(I - K)H^2$ is a closed subspace of L^2 since $T - (\lambda - l)I : H^2 \to L^2$ is an isomorphism and a homeomorphism by Corollary 3.7.

Lemma 3.14 *Let $V \in L^\infty$. For $\varepsilon > 0$, let X be a subspace of H^1 such that*

$$\int |\nabla u|^2 + V u^2 dx \leq (l - \varepsilon) \int u^2 dx \quad \textit{for all } u \in X. \tag{22}$$

Then $\dim X < \infty$.

Proof We begin by observing that $\int |\nabla u|^2 + Vu^2 dx$ and $\int u^2 dx$ are both continuous functions of u on H^1. Hence (22) also holds for all u in the closure of X. Thus we assume henceforth that X is a closed subspace of H^1. Consider a sequence $\{u_n\} \subset X$ such that $\|u_n\|_{H^1} = 1$ for all $n \in N$. We need only show that $\{u_n\}$ contains a subsequence which converges strongly in H^1. By passing to a subsequence we can immediately suppose, Weidmann [1] Theorem 4.25, that $u_n \rightharpoonup u$ weakly in H^1 for some element $u \in H^1$. If Pu denotes the orthogonal projection of u onto X in H^1, then

$$\|u - Pu\|_{H^1}^2 = \langle (I-P)u, u\rangle_{H^1} = \langle (I-P)u, u - u_n\rangle_{H^1} \to 0,$$

so $u = Pu \in X$.

By the definition of l, there exists $R > 0$ such that

$$V(x) \geq l - \frac{\varepsilon}{2} \quad \text{for almost all } |x| \geq R. \tag{23}$$

Then by the compactness of the Sobolev embedding of $H^1(B(0,R))$ into $L^2(B(0,R))$, it follows that

$$\int_{|x|\leq R} (u_n - u)^2 dx \to 0. \tag{24}$$

Also

$$\begin{aligned} & \frac{\varepsilon}{2}\int_{|x|\geq R}(u_n-u)^2dx + \int |\nabla(u_n-u)|^2 dx \\ \leq\ & \frac{\varepsilon}{2}\int_{|x|\geq R}(u_n-u)^2dx + (l-\varepsilon)\int (u_n-u)^2dx - \int V(u_n-u)^2dx, \qquad \text{by (22)} \\ =\ & \int_{|x|\geq R}[l - \frac{\varepsilon}{2} - V](u_n-u)^2dx + \int_{|x|\leq R}[l-\varepsilon-V](u_n-u)^2dx \\ \leq\ & (l + |V|_\infty)\int_{|x|\leq R}(u_n-u)^2dx \qquad \text{by (23).} \end{aligned}$$

It follows that $\int_{|x|\geq R}(u_n-u)^2dx \to 0$ and $\int |\nabla(u_n-u)|^2 dx \to 0$ as $n \to \infty$, which combined with (24) shows that $\|u_n - u\|_{H^1} \to 0$. This completes the proof.

We now come to the main result concerning the essential spectrum of S.

Theorem 3.15 *Let $V \in L^\infty$.*

(i) Then $\sigma_e(S) \subset [l, \infty)$.
(ii) If

$$\lim_{R\to\infty} ess \sup_{|x|\geq R} |V(x) - l| = 0, \quad \textit{then} \quad \sigma_e(S) = [l, \infty).$$

Remark 3.3 *If V is a bounded periodic function, then in general, $l > \inf \sigma_e(S)$ and $\sigma_e(S)$ is not an interval but a union of closed intervals, Berezin and Shubin [4] Chapter 3.4. The extra hypothesis in (ii) which gives a sharp conclusion will often be written more concisely as*

$$\lim_{|x|\to\infty} V(x) = l.$$

Observe that under this condition $\sigma_d(S) \neq \emptyset \iff \Lambda < l$.

Proof (i) We consider $\lambda < l$ and show that $\lambda \in \rho(S) \cup \sigma_d(S)$.

By Lemma 3.13 we know that $R(S-\lambda I)$ is a closed subspace of L^2. Hence $R(S - \lambda I) = [\ker(S - \lambda I)]^{\perp}$. Thus we see that if $\lambda \notin \sigma_p(S)$ then certainly $\lambda \in \rho(S)$.

Furthermore, if $u \in \ker(S - \lambda I)$,

$$\int |\nabla u|^2 + V u^2 dx = \lambda \int u^2 dx$$

and so, applying Lemma 3.14 to $\ker(S-\lambda I)$, we see that $\dim \ker(S-\lambda I) < \infty$.

Finally we chose any number $\xi < l$ and we show that $\sigma(S) \cap (-\infty, \xi]$ contains at most a finite number of points. Otherwise we would have a sequence $\{\lambda_i : i \in N\}$ such that

$$\lambda_i \in \sigma(S) \cap (-\infty, \xi] \quad \text{and } \lambda_i \neq \lambda_j \quad \text{for all } i, j \in N \text{ with } i \neq j.$$

By Theorem 3.10, we have $\lambda_i \geq \Lambda$ and by the first observation we must have that $\lambda_i \in \sigma_p(S)$ for all $i \in N$. Thus we can select a sequence of eigenvectors $\{e_i : i \in N\}$ such that

$$e_i \in \ker(S-\lambda_i I), \quad |e_i|_2 = 1 \quad \text{and} \quad \int e_i e_j dx = 0 \quad \text{for all } i, j \in N \text{ with } i \neq j.$$

Setting $X = span\{e_i : i \in N\}$, we must have $\dim X = \infty$ since the $\{e_i : i \in N\}$ are linearly independent.. On the other hand, for any $u \in X$, there exist a finite number of real numbers $c_1, ..\ \ c_k$ such that

$$u = \sum_{i=1}^{k} c_i e_i.$$

Therefore $u \in H^2$ and

$$\begin{aligned}\int |\nabla u|^2 + V u^2 dx &= \int (-\Delta u + V u) u dx = \int (Su) u dx \\ &= \int \left(\sum_{i=1}^{k} c_i \lambda_i e_i \right) \left(\sum_{j=1}^{k} c_j e_j \right) dx = \sum_{i,j=1}^{k} \lambda_i c_i c_j \int e_i e_j dx \\ &= \sum_{i=1}^{k} \lambda_i c_i^2 \le \xi \sum_{i=1}^{k} c_i^2 = \xi \int u^2 dx.\end{aligned}$$

Since $\xi < l$ the Lemma 3.14 shows that $\dim X < \infty$, contradicting the earlier conclusion. Thus we see that $\sigma(S) \cap (-\infty, \xi]$ cannot contain an infinite number of points and so the spectrum of S in the interval $(-\infty, \xi]$ is discrete for every $\xi < l$. This completes the proof of part (i).

(ii) Now we suppose that $\lim_{|x| \to \infty} V(x) = l$ and we shall show that $[l, \infty) \subset \sigma(S)$. Thus $[l, \infty) \subset \sigma_e(S)$ and since we already know from part (i) that $\sigma_e(S) \subset [l, \infty)$, this is sufficient to prove that $\sigma_e(S) = [l, \infty)$.

Choose any element $w \in D$ with $|w|_2 \neq 0$. Then, for $\lambda \ge l$, set

$$v_n(x) = n^{-\frac{N}{2}} w(\frac{x}{n}) \cos(\sqrt{\lambda - l} x_1) \text{ for } x \in I\!R^N.$$

It follows that

$$\begin{aligned}|v_n|_2^2 &= n^{-N} \int w(\frac{x}{n})^2 \cos^2(\sqrt{\lambda - l} x_1) dx \\ &= \begin{cases} \int w(y)^2 dy & \text{if } \lambda = l \\ \frac{1}{2} \int w(y)^2 \left\{ 1 + \cos(2\sqrt{\lambda - l} n y_1) \right\} dy & \text{if } \lambda > l \end{cases}.\end{aligned}$$

Hence

$$\begin{cases} |v_n|_2^2 = |w|_2^2 \text{ for all } n \in N & \text{if } \lambda = l \text{ and} \\ |v_n|_2^2 \to \frac{1}{2} |w|_2^2 \text{ as } n \to \infty & \text{if } \lambda > l, \end{cases} \tag{25}$$

by the Riemann-Lebesgue Lemma. Furthermore, recalling that for smooth functions φ and ψ,

$$\Delta(\varphi\psi) = (\Delta\varphi)\psi + \varphi(\Delta\psi) + 2\nabla\varphi \cdot \nabla\psi$$

we find that

$$(S - \lambda I) v_n(x) = -n^{-2} \Delta w(\frac{x}{n}) n^{-\frac{N}{2}} \cos(\sqrt{\lambda - l} x_1)$$

$$
\begin{aligned}
& +n^{-\frac{N}{2}} w(\frac{x}{n})(\lambda - l)\cos(\sqrt{\lambda - l}x_1) \\
& +2n^{-\frac{N}{2}} n^{-1}\partial_1 w(\frac{x}{n})\sqrt{\lambda - l}\sin(\sqrt{\lambda - l}x_1) \\
& +\{V(x) - \lambda\}\, n^{-\frac{N}{2}} w(\frac{x}{n})\cos(\sqrt{\lambda - l}x_1) \\
= \;& g_1(x) + g_2(x) + g_3(x)
\end{aligned}
$$

where

$$
\begin{aligned}
g_1(x) &= -n^{-2-\frac{N}{2}}\Delta w(\frac{x}{n})\cos(\sqrt{\lambda - l}x_1), \\
g_2(x) &= 2n^{-\frac{N}{2}} n^{-1}\partial_1 w(\frac{x}{n})\sqrt{\lambda - l}\sin(\sqrt{\lambda - l}x_1) \text{ and} \\
g_3(x) &= \{V(x) - l\}\, n^{-\frac{N}{2}} w(\frac{x}{n})\cos(\sqrt{\lambda - l}x_1).
\end{aligned}
$$

Now,

$$
\begin{aligned}
|g_1|_2^2 &= n^{-(N+4)}\int \{\Delta w(y)\}^2 \cos^2(\sqrt{\lambda - l}ny_1) n^N dy \\
&\leq n^{-4}\int \{\Delta w(y)\}^2 dy \text{ and} \\
|g_2|_2^2 &= 4n^{-(N+2)}\int \{\partial_1 w(y)\}^2 (\lambda - l)\sin^2(\sqrt{\lambda - l}ny_1) n^N dy \\
&\leq 4n^{-2}(\lambda - l)\int \{\partial_1 w(y)\}^2 dy
\end{aligned}
$$

so that $|g_1(x) + g_2(x)|_2 \to 0$ as $n \to \infty$. Also

$$
\begin{aligned}
|g_3|_2^2 &= n^{-N}\int \{V(x) - l\}^2 w(\frac{x}{n})^2 \cos^2(\sqrt{\lambda - l}x_1) dx \\
&= \int \{V(ny) - l\}^2 w(y)^2 \cos^2(\sqrt{\lambda - l}ny_1) dy \to 0 \text{ as } n \to \infty
\end{aligned}
$$

by dominated convergence since

$$\lim_{n\to\infty} V(ny) = l \text{ for almost all } y \in \mathbb{R}^N.$$

Thus we see that

$$|(S - \lambda I)v_n|_2 \to 0 \text{ as } n \to \infty$$

and, if $\lambda \in \rho(S)$, this implies that $|v_n|_2 \to 0$ as $n \to \infty$ by Lemma 3.2. In view of (25) we can conclude that $\lambda \notin \rho(S)$, as required.

This result furnishes some interesting information about the Helmholtz equation.

Corollary 3.16 *Let $V \equiv 0$. Then $\Lambda = l = 0, \sigma_p(S) = \emptyset$ and $\sigma_e(S) = \sigma(S) = [0, \infty)$.*

Proof By Theorem 3.8 , $\sigma_p(S) = \emptyset$ and clearly $l = 0$, so, by the previous result $\sigma_e(S) = [0, \infty)$. But it is also obvious that $\Lambda \geq 0$ and so, by Theorem 3.10, $\sigma(S) \subset [0, \infty)$.

Corollary 3.17 *Given any $\lambda \geq 0$, there exists $f \in L^2$ such that the Helmholtz equation (1) has no weak solution.*

Proof If u is a weak solution of (1) for an $f \in L^2$, we know from Theorem 2.12 that $u \in H^2$. Thus $f \in R(S - \lambda I)$ for the Schrödinger operator with potential $V \equiv 0$. But, since $\sigma_p(S) = \emptyset$, we know that $S - \lambda I$ is injective for all λ and consequently, $\lambda \in \rho(S) \iff R(S - \lambda I) = L^2$. By the previous corollary we have that $R(S - \lambda I) = L^2 \iff \lambda < 0$.

3.3 Properties of eigenfunctions

For the Schrödinger operator S with a potential $V \in L^\infty$, we obtain some additional properties of solutions of the homogeneous equation $Su = \lambda u$. We already know that such functions are quite smooth.

Theorem 3.18 *Let $V \in L^\infty$ and consider $u \in \ker(S - \lambda I)$ for some $\lambda \in I\!R^N$. Then*

$$u \in C \cap H^2 \cap W^{1,s} \textit{ for } 2 \leq s \leq \infty \quad \textit{and} \quad \lim_{|x| \to \infty} u(x) = 0.$$

Proof Setting $f = -Vu$ and using a bootstrap argument, this follows from Theorem 2.12 and Theorem 2.8.

Furthermore, below the essential spectrum of S the eigenfunctions even decay exponentially to zero at infinity in a uniform way.

Theorem 3.19 *Let $V \in L^\infty$ and choose $\xi < l$ where l is defined by (21). For any $\mu \in (0, \sqrt{l - \xi})$, there is a constant C, depending only on ξ and μ, such that*

$$|u(x)| \leq C \, |u|_\infty \, e^{-\mu |x|} \quad \textit{for all } x \in I\!R^N$$

provided that $u \in \ker(S - \lambda I)$ for some $\lambda \leq \xi$.

Proof Setting $r = |x|$, we note that

$$\Delta e^{-\mu r} = (e^{-\mu r})'' + \frac{N-1}{r}(e^{-\mu r})' = \left\{\mu^2 - \frac{N-1}{r}\mu\right\} e^{-\mu r} \quad \text{for } x \neq 0.$$

Since $0 < \mu^2 < l - \xi$, there exists $R = R(\xi, \mu) > 0$ such that

$$V(x) \geq \xi + \mu^2 \quad \text{for all } |x| \geq R$$

and so, for all $\lambda \leq \xi$, we also have that

$$\begin{aligned} V(x) &> \lambda \quad \text{and} \\ \lambda - V(x) + \mu^2 - \frac{N-1}{r}\mu &< 0 \quad \text{for all } |x| \geq R. \end{aligned}$$

Now set $C = e^{\mu R}$ and, for any $u \in \ker(S - \lambda I)\backslash\{0\}$ with $\lambda \leq \xi$, consider the function w defined by

$$w(x) = u(x) - C\,|u|_\infty\, e^{-\mu|x|} \quad \text{for all } x \in I\!\!R^N.$$

From the preceding theorem we know, that

$$w \in C \cap H^1 \quad \text{and} \quad \lim_{|x|\to\infty} w(x) = 0.$$

The definition of C ensures that

$$w(x) \leq 0 \quad \text{for all } |x| \leq R.$$

Thus, Gilbarg and Trudinger [6] Lemma 7.6,$w^+ \in C \cap H^1$, $\lim_{|x|\to\infty} w^+(x) = 0$ and $w^+ \equiv 0$ for $|x| \leq R$.

Let

$$\Omega = \left\{x \in I\!\!R^N : w^+ > 0\right\}.$$

Clearly Ω is open and $\Omega \subset I\!\!R^N \backslash \overline{B(0,R)} \equiv E(R)$. Let us suppose that $\Omega \neq \emptyset$. Then

$$\begin{aligned} &\int \left|\nabla(w^+)\right|^2 dx = \int_{E(R)} \nabla w \cdot \nabla(w^+) dx \\ =\ & -\int_{E(R)} (\Delta w) w^+ dx \quad \text{since } w \in H^2(E(R)) \text{ and } w^+ = 0 \text{ on } \partial E(R) \\ =\ & -\int_\Omega (\Delta w) w dx \\ =\ & \int_\Omega \left\{(\lambda - V)u + C\,|u|_\infty\, \Delta(e^{-\mu|x|})\right\} w dx \\ \leq\ & \int_\Omega \left\{\lambda - V + \mu^2 - \frac{N-1}{r}\mu\right\} C\,|u|_\infty\, e^{-\mu|x|} w dx \end{aligned}$$

since $\lambda - V(x) \leq 0$ and $u(x) > C\,|u|_\infty\, e^{-\mu|x|}$ on Ω. But $w > 0$ on $\Omega \subset E(R)$ and R was chosen so that

$$\lambda - V + \mu^2 - \frac{N-1}{r}\mu < 0 \text{ on } E(R).$$

Thus we see that

$$0 \leq \int \left|\nabla(w^+)\right|^2 dx < 0$$

if $\Omega \neq \emptyset$. Thus we must have that $\Omega = \emptyset$ and $w \leq 0$ on $I\!R^N$. Hence $u(x) \leq C\,|u|_\infty\, e^{-\mu|x|}$ for all $x \in I\!R^N$.

Replacing u by $-u$ completes the proof.

We already know that $\Lambda \in \sigma_d(S)$ whenever $\Lambda < l$. As a final result we show that in this case Λ is a simple eigenvalue with a strictly positive eigenfunction.

Theorem 3.20 *Let $V \in L^\infty$ with $\Lambda < l$ where Λ and l are defined by (20) and (21) respectively. Then there is an element $\psi \in C \cap H^2$ such that*

$$\psi(x) > 0 \quad \textit{for all } x \in I\!R^N \quad \textit{and } \ker(S - \Lambda I) = \textit{span}\,\{\psi\}.$$

Remark 3.4 *Of course ψ also has the additional regularity ensured by Theorem 3.18 as well as the exponential decay given by the preceding theorem.*

Proof From Theorems 3.10 and 3.15 we know that $\Lambda \in \sigma_d(S)$. Let φ be any element of $\ker(S - \Lambda I)$ with $|\varphi|_2 = 1$. Then

$$\int |\nabla\varphi|^2 + V\varphi^2 dx = \int (S\varphi)\varphi dx = \Lambda.$$

Setting $\psi = |\varphi|$, we have that $\psi \in H^1$ and $\int |\nabla\psi|^2\, dx = \int |\nabla\varphi|^2\, dx$. Hence

$$\int |\nabla\psi|^2 + V\psi^2 dx = \Lambda \quad \text{and } |\psi|_2 = |\varphi|_2 = 1.$$

By Lemma 3.9(3), it follows that $\psi \in \ker(S - \Lambda I)$ and then by Theorem 3.18 we know that $\psi \in C \cap W^{1,s}$ for $2 \leq s \leq \infty$. The exponential decay of ψ to zero as $|x| \to \infty$ follows from Theorem 3.19. But for any $\lambda \in I\!R$,

$$-\Delta\psi(x) = \lambda\psi(x) + \{\Lambda - V(x) - \lambda\}\,\psi(x) \quad \text{a.e. on } I\!R^N$$

and we can choose $\lambda < 0$ such that

$$\{\Lambda - V(x) - \lambda\} \geq 1 \quad \text{ a.e. on } I\!R^N.$$

Setting $f(x) = \{\Lambda - V(x) - \lambda\}\, \psi(x)$, it follows that $f \in L^2 \cap L^\infty$ with $f(x) \geq 0$ a.e. on $I\!R^N$ and $|f|_2 \geq |\varphi|_2 = 1$. From Theorem 2.9, $\psi(x) > 0$ for all $x \in I\!R^N$.

To complete the proof we need only show that $span\,\{\psi\} = \ker(S - \Lambda I)$. However if this were not the case then there would be an element $\varphi \in \ker(S - \Lambda I)$ such that $|\varphi|_2 = 1$ and $\int \varphi\psi dx = 0$. But the first part of the proof shows that $\varphi \in C$ and that, for every $x \in I\!R^N, \varphi(x) \neq 0$ since $|\varphi(x)| > 0$ for all $x \in R^N$. Thus $\varphi\psi \in C$ and for all $x \in I\!R^N, \varphi(x)\psi(x) \neq 0$, contradicting the fact that $\int \varphi\psi dx = 0$. This completes the proof.

4 References

References

[1] Weidmann, J. : Linear Operators in Hilbert Space, Springer Graduate Text in Mathematics, Springer-Verlag, New York, 1980

[2] Rudin, W. : Functional Analysis, McGraw-Hill, New york, 1973

[3] Lebedev, N.N. : Special Functions and their Applications, Prentice Hall, Englewood Cliffs, 1965 (Dover edition 1972)

[4] Berezin, F.A. and Shubin, M.:A: : The Schrödinger Equation, Kluwer Academic Publishers, Dordrecht, 1991

[5] Brézis, H. : Analyse Fonctionnelle, Masson, Paris, 1983

[6] Gilbarg, D. and Trudinger, N.S. : Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin, 1977

[7] Yosida, K. : Functional Analysis, 6th edition, Springer-Verlag, Berlin, 1980

[8] Titchmarsh, E.C. : Introduction to the Theory of Fourier Integrals, Oxford University Press, Oxford, 1937

[9] Deimling, K. : Nonlinear Functional Analysis, Springer-Verlag, New York,1985

[10] Adams, R.A. : Sobolev Spaces, Academic Press, New York, 1975